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DIAGRAM GROUPS, BRAID GROUPS, AND ORDERABILITY

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ABSTRACT

We prove that all diagram groups (in the sense of Guba and Sapir) are left-orderable. The proof is in two steps: firstly, it is proved that all diagram groups inject in a certain braid group on infinitely many strings, and secondly, this group is then shown to be left-orderable.

Keywords: diagram group, braid group, Thompson group, orderable group

1. Introduction

1.1. *Statement of the result*

Diagram groups were defined by Meakin and Sapir in 1993, and we shall recall this definition below. The motivating example was the well-known R. Thompson's group F, i.e. the group of all PL homeomorphisms of the interval $[0,1]$ whose break points are dyadic integers, and whose values in the break points are also dyadic integers. Many strong theorems about the structure of diagram groups are now known, mostly due to Guba and Sapir [5, 6, 7], but also to Kilibidara [10] and Farley [3]. Some of these results were new even for Thompson's group F. One of the major questions left open was whether all diagram groups are bi-orderable, i.e. can be equipped with a total ordering $<$ which is invariant under multiplication on the left and on the right ($g < h \Longrightarrow (ag < ah$ and $ga < ha)$ for all elements a, g, h). This is a natural question, since the group F (and indeed the supergroup of *all* PL homeomorphisms of the interval $[0,1]$) is quite easily shown to be bi-orderable.

The aim of the present paper is, unfortunately, not to answer this question, but at least to settle a slightly weaker one:

Theorem 1.1 (Main theorem). *All diagram groups are left-orderable, i.e. they can be equipped with a total ordering which is invariant under multiplication on the left.*

We remark that left-orderability is equivalent to right-orderability, and is a strictly stronger condition than torsion-freeness. For more background on the importance of being orderable, see e.g. the open questions section of [5].

The proof is in two steps, which are of interest in their own right. First we prove that diagram groups are subgroups of a certain braid group $\mathcal{B}$ on infinitely many strings, and then we prove that the braid group $\mathcal{B}$ is left-orderable. Since subgroups of left-orderable groups are left-orderable, this implies the main theorem.

The paper is organized as follows: in the rest of section 1 we recall the definition of diagram groups, and then define the braid group $\mathcal{B}$. In section 2 we prove that all diagram groups are subgroups of $\mathcal{B}$. Finally in section 3 we prove that this braid group $\mathcal{B}$ is left-orderable.

Remark After this paper was written, Mark Sapir informed me that he has proved with V. Guba that diagram groups are bi-orderable [8], by embedding them in a certain universal diagram group which is then proved to be bi-orderable.

1.2. *Diagram groups*

In this section we explain very briefly the definition and properties of diagram groups which are useful for our purposes. A much more detailed account can be found in [5].

Let $\mathcal{P} = \langle \Sigma \,|\, \mathcal{R} \rangle$ be any semigroup-presentation; that is, Σ is a finite set of generators, and the set $\mathcal{R}$ consists of finitely many word rewriting rules of the form $q_i \to q_i'$, where q_i and q_i' are words in the alphabet Σ. For simplicity we suppose that if $(q_i \to q_i') \in \mathcal{R}$, then $(q_i' \to q_i) \notin \mathcal{R}$. The number of elements of $\mathcal{R}$ is denoted r. These generators and relations define a semigroup: the elements are all finite words in the letters Σ, modulo the equivalence relation generated by the word rewriting rules in $\mathcal{R}$ (and their inverses), and multiplication is given by concatenation of words.

If w' and w'' are words in the letters Σ, then we define a (w', w'')-diagram to be a finite sequence of words in the letters Σ, starting with w' and terminating with w'', where each word is obtained from its predecessor by a single application of one of the word rewriting rules in $\mathcal{R}$ or their inverses. Every step of the sequence of word replacements is called a *cell* of the diagram. Thus a typical (w', w'')-diagram (with say n cells) takes the shape

$$w' \xrightarrow{\Delta_1} w_1 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{n-1}} w_{n-1} \xrightarrow{\Delta_n} w''$$

where each cell is either of the form $p'qp'' \xrightarrow{\Delta} p'q'p''$ or of the form $p'q'p'' \xrightarrow{\Delta^{-1}} p'qp''$, where $(q \xrightarrow{\Delta} q') \in \mathcal{R}$. We say q is the *top* and q' the *bottom* of the cell Δ, a fact which is written formally as $\mathbf{top}(\Delta) = q$ and $\mathbf{bot}(\Delta) = q'$. Similarly, for the cell Δ^{-1} we have $\mathbf{top}(\Delta^{-1}) = q'$ and $\mathbf{bot}(\Delta^{-1}) = q$.

We define an equivalence relation on the set of (w', w'')-diagrams, which is generated by the requirements that distant cells should commute, and that dipoles can be inserted or deleted anywhere in the sequence. More precisely, we define the two diagrams

$$w' \longrightarrow \ldots \longrightarrow p_1 q_1 p_2 q_2 p_3 \xrightarrow{\Delta_1} p_1 q_1' p_2 q_2 p_3 \xrightarrow{\Delta_2} p_1 q_1' p_2 q_2' p_3 \longrightarrow \ldots \longrightarrow w'' \quad \text{and}$$
$$w' \longrightarrow \ldots \longrightarrow p_1 q_1 p_2 q_2 p_3 \xrightarrow{\Delta_2'} p_1 q_1 p_2 q_2' p_3 \xrightarrow{\Delta_1'} p_1 q_1' p_2 q_2' p_3 \longrightarrow \ldots \longrightarrow w''$$

to be equivalent. Similarly, we define the two diagrams

$$w' \longrightarrow \ldots \longrightarrow w_j \xrightarrow{\Delta} w_{j+1} \xrightarrow{\Delta^{-1}} w_j \longrightarrow \ldots \longrightarrow w'' \quad \text{and}$$
$$w' \longrightarrow \ldots \longrightarrow w_j \longrightarrow \ldots \longrightarrow w''$$

to be equivalent.

We say a (w', w'')-diagram is *reduced* if it is not equivalent to diagram with fewer cells, or equivalently, if it has no subsequence of the form $\xrightarrow{\Delta} w_j \xrightarrow{\Delta_{j+1}} \ldots \xrightarrow{\Delta_k} w_k \xrightarrow{\Delta^{-1}}$, where $\mathbf{top}(\Delta_{j+1}), \ldots, \mathbf{top}(\Delta_k)$ do not contain any letters which were created by Δ, the jth cell of the sequence. Every equivalence class of (w', w'')-diagrams is represented by a reduced diagram which is unique up to applications of the commutation relation [10].

Definition 1.1. *Let w be a word in the letters Σ. Then the* diagram group $\mathcal{D}(\mathcal{P}, w)$ *is the group of all equivalence classes of (w, w)-diagrams.*

1.3. *The braid group $\mathcal{B}$*

We recall that the classical braid group B_n can be defined in two ways. Let Π_n be an arbitrary configuration of n distinct unordered points, called *particles*, in the interior of the closed disk D^2. Then B_n is the mapping class group of the pair (D^2, Π_n), i.e. the group of isotopy classes of homeomorphisms of the disk which fix the boundary and permute the particles. Alternatively, one can define B_n as the group of isotopy classes of particle dances, a notion which will be made precise below.

Analogously, we have two ways of thinking about the following group $\mathcal{B}$ which could be loosely described as a group of braids with infinitely many strings which are all allowed to move simultaneously:

Definition 1.2. *Let Π be the set of integral points in the first quadrant of the plane: $\Pi := \{(x, y) \mid x, y \in \mathbb{N} \setminus \{0\}\}$. These points are called the* particles. *We define the group $\mathcal{B}$ to be the mapping class group of the pair $(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}, \Pi)$, i.e. the group homeomorphisms of the first quadrant $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$ of the plane $\mathbb{R} \times \mathbb{R}$ which fix the boundary $\partial(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0})$ and permute the particles Π.*

Often, the most convenient way to describe an element of $\mathcal{B}$ is in terms of particle dances. Let $\mathcal{S}$ be the totally disconnected topological space consiting of countably many points. A *particle dance*, is a continuous mapping $\varphi\colon \mathcal{S} \times [0,1] \to \mathbb{R}_+ \times \mathbb{R}_+$ satisfying three conditions. Firstly that $\varphi(\mathcal{S} \times \{0\}) = \varphi(\mathcal{S} \times \{1\}) = \Pi$. Secondly that $\varphi(\,.\, \times \{t\})\colon \mathcal{S} \to \mathbb{R}_+ \times \mathbb{R}_+$ is injective for every $t \in [0,1]$. And thirdly we require the following tameness condition: for every compact subset K of $\mathbb{R}_+ \times \mathbb{R}_+$ only finitely many particles may intersect K during the dance; more formally, we want that

$$|\{s \in \mathcal{S}\ ;\ \exists x \in [0,1] \text{ such that } \varphi((s,x)) \in K\}| < \infty.$$

A *braid* is a homotopy class of particle dances; that means, two particle dances φ and ψ define the same braid if and only if there exists a continuous family of particle dances relaying φ with ψ. We can define a group $\mathcal{B}'$ as the set of all braids, where multiplication is given by concatenation of particle dances. Braids, in this sense, are useful for describing elements of $\mathcal{B}$, because of the following

Lemma 1.1. *There is a natural monomorphism $\mathcal{B}' \to \mathcal{B}$.*

Proof The proof is very similar to the proof of the analogue statement for classical braid groups, which is e.g. explained in [1] and [2]. We shall only sketch it here, and leave it to the reader to verify that our tameness condition for particle dances suffices to guarantee that the obvious analogue of the classical construction works in our context.

Any particle dance can be extended to a boundary-fixing isotopy of the identity map id: $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$. Such an isotopy terminates at a representative of an element of the mapping class group $\mathcal{B}$. Moreover, any two such isotopies for the same particle dance terminate at isotopic homeomorphisms. (We remark that our monomorphism is in fact an isomorphism, but we shall not use this result.) $\square$

We have to set up some more notation which will be useful for proving both the embedding property of diagram groups in $\mathcal{B}$ and the left-orderability of $\mathcal{B}$.

We define a diagram D, called the *trivial curve diagram* as follows. The diagram consists of infinitely many closed straight line segments in the first quadrant of the plane, exactly one for each particle. These line segments are oriented; they are starting all on one and the same point, namely a point $(x, 0)$ in the boundary of the quadrant, where x is an arbitrary irrational positive number, and they terminate on the particles. Thus every particle is connected by exactly one line segment to the point $(x, 0)$, the line segments do not intersect the particles except in their endpoints, and they emanate in a startlike fashion from the point $(x, 0)$.

We denote the particles (the elements of Π) by $\pi_{i,j}$ $(i, j \in \mathbb{N} \setminus \{0\})$, and the line segment from $(x, 0)$ to $\pi_{i,j}$ by $D_{i,j}$.

Definition 1.3. *A* curve diagram *is any diagram in the first quadrant which is the image of D under a homeomorphims φ of the first quadrant with $\varphi(\Pi) = \Pi$ which fixes the boundary of the first quadrant. Thus any curve diagram consists of countably many arcs, which are disjoint except in the point $(x, 0)$, each connecting the point $(x, 0)$ to one of the particles.*

Observation Every element of $\mathcal{B}$ gives, in a natural way, rise to an isotopy class of curve diagrams. Thus it makes sense to speak of "a curve diagram of β" for any $\beta \in \mathcal{B}$.

Next we define a notion of "non-wigglyness" of curve-diagrams. First we notice that the one-dimensional integer lattice subdivides the first quadrant into infinitely many squares (which have the particles at their corners). The intersection of any curve diagram with any such square consists, at least after an arbitrarily small perturbation, of a (finite or infinite) number of arcs, which are properly embedded in the square. Now we define a curve diagram D' to be *reduced* if none of these arcs have both of their endpoints in the interior of the same side of the square.

Every curve diagram D' is isotopic to a reduced one, and this reduced version of D' is combinatorially unique. For details about such reduction-procedures see e.g. [12] or [4].

By a slight abuse of notation, we shall denote by $\beta(D)$ any *reduced* curve diagram of β. We shall think of this as the result of acting on D by a particle dance representing β; we stress that the diagram $\beta(D)$ is indeed well-defined up to isotopies of the first quadrant which fix the one-dimensional integer lattice setwise.

2. All diagram groups embed in the braid group $\mathcal{B}$

Proposition 2.1. *Let $\mathcal{P} = \langle \Sigma \,|\, \mathcal{R} \rangle$ be any semigroup-presentation, and w a word in the letters Σ. Then there exists a monomorphism Φ from the diagram group $\mathcal{D}(\mathcal{P}, w)$ to the braid group $\mathcal{B}$.*

Proof The proof will occupy the rest of this section. First we observe that it suffices to consider the special case where all the words appearing in the set of relations $\mathcal{R}$ have length at least 2 – for instance, the semigroup-presentation $\mathcal{P} = \langle \{\sigma_1, \sigma_2\} \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_1 \rangle$ would be excluded. To see why, we consider the diagram group $\mathcal{P} = \langle \Sigma \,|\, \mathcal{R}' \rangle$ which has by definition the same generating set as $\mathcal{P}$, and whose relations $\mathcal{R}'$ are obtained from the relations $\mathcal{R}$ by replacing every letter by its square – e.g. in the above example we would have $\mathcal{P}' = \langle \{\sigma_1, \sigma_2\} \,|\, \sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1 = \sigma_1\sigma_1 \rangle$. The homomorphism of monoids $\mathcal{P} \to \mathcal{P}'$ which is defined by sending every generator to its square induces a homomorphism of diagram groups $\mathcal{D}(\mathcal{P}, w) \to \mathcal{D}(\mathcal{P}', w')$, where the word w' is obtained by replacing every letter of w by its square. This homomorphism is injective, because it sends reduced diagrams to reduced diagrams. Thus it suffices to prove that the group $\mathcal{D}(\mathcal{P}', w')$ is left-orderable, and among the relations of $\mathcal{P}'$ we have indeed no words of length 1.

Thus for the rest of the proof we assume that the words appearing in the relations of $\mathcal{P}$ have length at least two. Every element of $\mathcal{D}(\mathcal{P}, w)$ is represented by a (w, w)-diagram over $\mathcal{P}$, that is, a sequence of words which starts and terminates with w, such that each element of the sequence is obtained from the preceding one by a replacement $p'qp'' \to p'q'p''$, where $q \equiv q'$ is one of the relations from $\mathcal{R}$.

We shall define the braids $\Phi(\Delta)$, where Δ denotes any cell (i.e. one step in our word rewriting sequence) $p'qp'' \xrightarrow{\Delta} p'q'p''$. This mapping Φ should satisfy the following conditions:

(a) (Dipoles are sent to the trivial braid) $\Phi(\Delta)\,(\Phi(\Delta^{-1})) = 1_{\mathcal{B}}$

(b) If

$$\begin{aligned} p_1q_1p_2q_2p_3 &\xrightarrow{\Delta_1} p_1q_1'p_2q_2p_3 \xrightarrow{\Delta_2} p_1q_1'p_2q_2'p_3 \\ \text{and } p_1q_1p_2q_2p_3 &\xrightarrow{\Delta_2'} p_1q_1p_2q_2'p_3 \xrightarrow{\Delta_1'} p_1q_1'p_2q_2'p_3, \end{aligned}$$

then in the braid group $\mathcal{B}$ the following equality holds: $\Phi(\Delta_1)\Phi(\Delta_2) = \Phi(\Delta_2')\Phi(\Delta_1')$.

(c) (Injectivity) If $w \xrightarrow{\Delta_1} \ldots \xrightarrow{\Delta_n} w$ is a reduced diagram representing an element of $\mathcal{D}(\mathcal{P}, w)$, and $n \neq 0$, then in the braid group $\mathcal{B}$ we have $\Phi(\Delta_1)\ldots\Phi(\Delta_n) \neq 1$.

In order to clarify condition (b) we remark that two "distant" cells are not exactly sent to commuting braids – in fact, condition (b) will turn into an exact commutation condition only in the special case that $|\mathbf{top}(\Delta_1)| = |\mathbf{bot}(\Delta_1)|$ (where $|\,.\,|$ denotes the length of a word).

Constructing braids $\Phi(\Delta)$ for every type of cell Δ (i.e. for every element of $\mathcal{R}$) satisfying conditions (a), (b), and (c) is sufficient in order to prove proposition 2.1. Thus the rest of the proof is divided into two parts: firstly, the construction, and secondly, the proof that this construction satisfies (a), (b) and (c).

The idea of the construction is to send a cell $p'qp'' \xrightarrow{\Delta} p'q'p''$ to a product of three braids: the first one is an "extremely complicated" braid which moves only particles in the columns corresponding to the subword q; more precisely, it is supported in the region $(|p'|, |p'| + |q| + 1) \times \mathbb{R}_+$ of $\mathbb{R}_+ \times \mathbb{R}_+$. The second braid redistributes the particles in those same columns over $|q'|$ columns, while moving all particles in columns number $|p'| + |q| + k$ ($k \geqslant 1$) rigidly to the right or left. The third braid is another "very complicated" braid, this time moving particles in the columns corresponding to the subword q'.

It is plausible that if the "very complicated" braids are chosen sufficiently complicated and sufficiently different from each other, then the injectivity condition (c) will be satisfied. The details of the construction and the proof, however are quite involved. Our braids $\Phi(\Delta)$ will be products of certain other braids, which we define first, and which will be called $\beta_{i,j}$, ϵ_i, and $\delta_{i,j}$.

We recall that the set $\mathcal{R}$ has r elements, all of them being relations of the form $q_i \to q_i'$, where q_i and q_i' are words in the alphabet Σ. We define the *width W of the presentation $\mathcal{P}$* to be $W := \sum_{i=1}^{r} |q_i| + |q_i'|$. So W is the sum of the lengths of all the words appearing in the relations of the semigroup presentation.

We start by defining a family $\alpha_1, \ldots, \alpha_W$ of pseudo-Anosov elements of the classical three-string braid group B_3, which we identify with the mapping class group of a disk with three punctures lined up on a *vertical* axis. In order to make this definition we need some facts concerning the space of measured foliations, which can be found in [9]. We choose arbitrarily some pseudo-Anosov elements $\alpha_1', \ldots, \alpha_W'$ of B_3 such that the stable and unstable measured foliations of each automorphism is distinct from both the stable and unstable foliation of all other automorphisms. After replacing each element by a high enough power, we can assume that they are all pure braids. We can also assume that the measured foliations are reduced, in the sense that there are no bigons enclosed between any leaves of the foliation and the vertical line or one of the three horizontal lines through the punctures. Let us consider, for each of the three punctures π_i ($i \in \{1,2,3\}$) and for each automorphism α'_j, the leaves of the stable and of the unstable foliation of α'_j with endpoint at π_i. We denote these leaves $S_{i,j}$ and $U_{i,j}$, respectively. Since the $2W$ foliations are pairwise distinct, there exist finite terminal segments $S^{\rm tm}_{i,j}$ and $U^{\rm tm}_{i,j}$ of these leaves such that for every fixed $i \in \{1,2,3\}$ the curves $S^{\rm tm}_{i,1}, \ldots, S^{\rm tm}_{i,W}, U^{\rm tm}_{i,1}, \ldots, U^{\rm tm}_{i,W}$ intersect the vertical axis the same number of times but are pairwise non-isotopic (in the sense that they have different sequences of intersection with the four segments of the vertical line). Next, we recall that a sufficiently high power of α'_j sends the whole space of measured foliations except for an arbitrarily small neighbourhood of the unstable foliation of α'_j into an arbitrarily small neighbourhood of the stable foliation. Therefore, if we take sufficiently high powers of the automorphisms $\alpha'_1, \ldots, \alpha'_W$, we obtain automorphisms $\alpha_1, \ldots, \alpha_W$ satisfying the following property:

Property If c is a reduced simple arc in the three-times punctured disk starting on the boundary of the disk and ending in the ith puncture ($i \in \{1,2,3\}$), and if, for some $j_0 \in \{1, \ldots, W\}$, the curve c does not have a terminal segment isotopic as a reduced curve to the arc $U^{\rm tm}_{i,j_0}$, then the action of α_{j_0} sends c to an arc which has a terminal segment isotopic to $S^{\rm tm}_{i,j_0}$. (In particular, this holds if c is either disjoint from the vertical line through the punctures, or has a terminal segment which is isotopic to one of the arcs $S^{\rm tm}_{i,j}$ or $U^{\rm tm}_{i,j}$ with $j \neq j_0$.) Similarly, $\alpha_{j_0}^{-1}$ sends any curve not ending with $S^{\rm tm}_{i,j_0}$ to a curve with a terminal segment which is isotopic to $U^{\rm tm}_{i,j_0}$.

We define, for every $i \in \mathbb{N} \setminus \{0\}$, the *ith column* of the first quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ to consist of all particles with coordinates (i, n) for any $n \in \mathbb{N} \setminus \{0\}$.

We are now ready to define the braids $\beta_{i,j}$. For any i in $\mathbb{N} \setminus \{0\}$ and any j in $\{1, \ldots, W\}$ we define the braid $\beta_{i,j}$ by a simultaneous dance of all the particles in the ith column, where each triple $\{\pi_{i,3k+1}, \pi_{i,3k+2}, \pi_{i,3k+3}\}$ ($k \in \mathbb{N}$) of particles performs the braid α_j. All particles outside the ith column are fixed by the braid $\beta_{i,j}$. This is illustrated in figure 1(a).

We have to define, for every $i \in \mathbb{N} \setminus \{0\}$, another braid $\epsilon_i \in \mathcal{B}$, which can be described as an "expansion" around the ith column (c.f. figure 1(b)). A representative particle dance is as follows:

- All particles $\pi_{j,k}$ with $j < i$ are fixed,
- any particle $\pi_{j,k}$ with $j \geqslant i+1$ is moved along a straight line to the right, to the point with coordinates $(j+1, k)$. Afterwards,
- the particles in the ith column, i.e. the particles $\pi_{j,k}$ with $j = i$ are distributed evenly over two columns, namely the ith column, and the newly vacated $i+1$st column. For the sake of definiteness we give a precise description: the particles

$\pi_{i,k}$, with k even, move in a straight line one step to the right, to the position $(i+1, k)$. Following this, the particles in the ith and $i+1$st column move down by simultaneous linear movements, until all integer lattice points are again occupied by a particle.

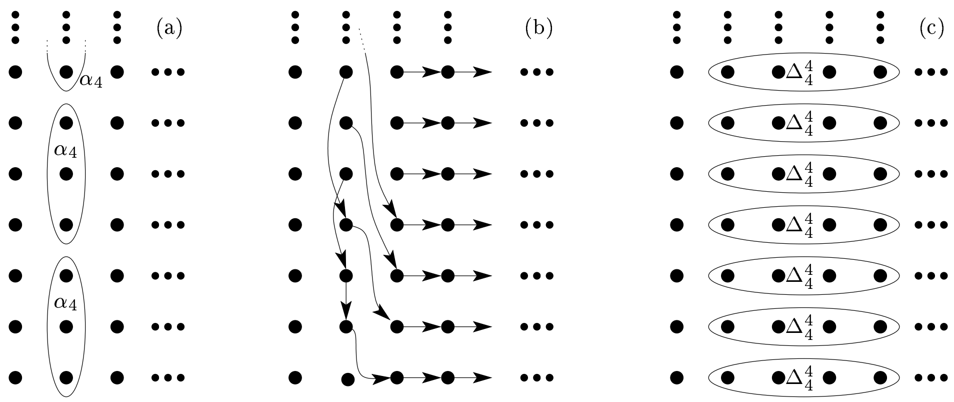

Figure 1: (a) The braid $\beta_{2,4}$ (second column, braid α_4) (b) The braid ϵ_2 (expanding to the right of the second column) (c) The braid $\delta_{2,5}$ (tying together columns 2 to 5)

Finally, we need a third type of braid, called δ_{k_0,k_1}, for any k_0, k_1 in $\mathbb{N}\setminus\{0\}$ with $k_0 < k_1$. We shall think of this braid as "horizontally tying together" the columns number $k_0, k_0+1, \ldots, k+1$. In each row we imagine a horizontal ellipse containing the punctures in columns number $k_0, \ldots, k_1$. In all these ellipses simultaneously, we perform the fourth power of a Garside half twist Δ (i.e. two full twists).

We are now ready to define the braids $\Phi(\Delta)$. These braids will be products of braids of the type $\beta_{i,j}^{\pm 1}$, $\epsilon_i^{\pm 1}$ and $\delta_{k,l}^{\pm 1}$. We start by performing the following quite radical operation: from the words $q_1, q_1', \ldots, q_r, q_r'$ we define new words $\widetilde{q}_1, \widetilde{q}_1', \ldots, \widetilde{q}_r, \widetilde{q}_r'$ in the letters $\{1, 2, \ldots, W\}$ in the following way: $\widetilde{q}_1 := 1\,2\,\ldots\,(|q_1|)$, $\widetilde{q}_1' := (|q_1|+1)\,\ldots\,(|q_1|+|q_1'|)$, and so on until $\widetilde{q}_r' := (W - |q_r'| + 1)\,\ldots\,W$.

Example 2.1. If $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ and $\mathcal{R} = \{\sigma_3\sigma_2 \equiv \sigma_1\sigma_3,\ \sigma_1\sigma_2\sigma_3 \equiv \sigma_2\sigma_1,\}$, then $r = 2$, $W = 9$ and $\widetilde{q}_1 = 1\,2$, $\widetilde{q}_1' = 3\,4$, $\widetilde{q}_2 = 5\,6\,7$, $\widetilde{q}_2' = 8\,9$.

Now if Δ is a cell of the form $p'q_ip'' \xrightarrow{\Delta} p'q_i'p''$, and $\widetilde{q}_i = \lambda \ldots \mu$ and $\widetilde{q}_i' = \lambda' \ldots \mu'$ (with $\lambda, \mu, \lambda', \mu' \in \mathbb{N}\setminus\{0\}$, $\lambda < \mu < \lambda' < \mu'$), then we define

$$\begin{aligned}\Phi(\Delta) \quad := \quad & \beta_{|p'|+1,\lambda} \cdot \beta_{|p'|+2,\lambda+1} \cdot \ldots \cdot \beta_{|p'|+|q_i|,\mu} \cdot \\ & \delta_{|p'|+1,|p'|+|q_i|} \cdot \epsilon_{|p'|+1}^{|q_i'|-|q_i|} \cdot \delta_{|p'|+1,|p'|+|q_i'|} \cdot \\ & \beta_{|p'|+1,\lambda'} \cdot \beta_{|p'|+2,\lambda'+1} \cdot \ldots \cdot \beta_{|p'|+|q_i'|,\mu'}\end{aligned}$$

This means the following. We have a composition of five particle dances. The first one consists of some pseudo-Anosov braids in the columns number $|p'|+1, \ldots, |p'|+|q_i|$, i.e. in exactly the positions corresponding to the subword q_i in $p'q_ip''$ (i.e. to $\mathbf{top}(\Delta)$). In the second dance we tie the columns corresponding to our subword together – we recall that this subword has by hypothesis length at least two, as assumed in the first paragraph of the proof. In the third dance, we expand or retract about the $|p'|+1$st column; more precisely, if $\mathbf{bot}(\Delta)$ is x letters longer than $\mathbf{top}(\Delta)$, then we have to create exactly x new columns; similarly, if $\mathbf{bot}(\Delta)$ is shorter than $\mathbf{top}(\Delta)$, then we have to get rid of the appropriate number of columns. In the fourth dance, we tie the newly augmented or reduced set of columns together once more (again, there are at least two of them), and in the very end we perform another β-type movement, just like the first dance, only corresponding to the subword q_i' rather than q_i.

Example 2.1 (continued) In the above mentioned example, the word replacement $\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \to \sigma_2\sigma_2\sigma_1\sigma_2$ would be sent to the braid $\beta_{2,5}\,\beta_{3,6}\,\beta_{4,7}\cdot\delta_{2,4}\cdot\epsilon_2^{-1}\cdot\delta_{2,3}\cdot\beta_{2,8}\,\beta_{3,9}$.

The braid $\Phi(\Delta^{-1})$ is, of course, defined by the formula $\Phi(\Delta^{-1})=(\Phi(\Delta))^{-1}$. This completes the definition of the mapping Φ.

It remains to check that Φ satisfies conditions (a), (b) and (c). The dipoles-condition (a) holds by construction, the distant-cells condition (b) is very easy to check, and the only condition whose verification needs work is the injectivity condition (c).

We recall that our given representative $w \xrightarrow{\Delta_1} \ldots \xrightarrow{\Delta_n} w$ was supposed reduced; so there is no subsequence $\xrightarrow{\Delta} w_j \xrightarrow{\Delta_{j+1}} \ldots \xrightarrow{\Delta_k} w_k \xrightarrow{\Delta^{-1}}$, where the cells $\Delta_{j+1},\ldots,\Delta_k$ do not affect any letters created by Δ, the jth cell of the diagram.

We suppose that $w \xrightarrow{\Delta_1} w_1 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_i} w_i$, with $i<n$, is an initial segment of the given representative. For every letter of w_i there are two possibilities: either it comes directly from the initial word w, and was not involved in any of the word replacements $\Delta_1,\ldots,\Delta_i$, or it arises as a letter in $\mathbf{bot}(\Delta_j)$ for a unique $j\leqslant i$. In the first case, if say the kth letter of w_i comes directly from the initial word w, then during the dance $\Phi(\Delta_1\cdot\ldots\cdot\Delta_i)$ the particles in the kth column were not involved in any pseudo-Anosov type dance (coming from a braid $\beta_{i,j}$) or a Garside-twist type dance (coming from a braid δ_i), they were only subject to horizontal shifts (coming from braids ϵ_i in columns to their left). In the second case, however, if the kth letter of w_i arose as a letter in $\mathbf{bot}(\Delta_j)$, then the particles which are now in the kth column recently performed a β-type dance coming from the last one of the five factors of $\Phi(\Delta_j)$, after which they were only moved rigidly by horizontal shifts. In this case, let us suppose, for definiteness, that the last pseudo-Anosov action on the particles which are now in the kth column was by the automorphism $\alpha_{j_0}^{\pm1}$ ($j_0\in\{1,\ldots,W\}$) acting simultaneously on all triples of particles at height $3n+1$, $3n+2$, and $3n+3$ ($n\in\mathbb{N}$).

Claim (a) *If the kth letter of w_i comes directly from the initial word w, then no curve of the curve diagram of $\Phi(\Delta_1\cdot\ldots\cdot\Delta_i)$ intersects the vertical line $\{k\}\times\mathbb{R}_+$ twice in a row without intersecting another vertical line $\{l\}\times\mathbb{R}_+$ ($l\in\mathbb{N}\setminus\{0,k\}$) in* *the mean time.*

(b) *If, on the other hand, the kth letter of w_i arises in* $\mathbf{bot}(\Delta_j)$ *as described above, then the curve of the curve diagram of* $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i)$ *ending at the position* $(k, 3n+r)$ $(n \in \mathbb{N},\ r \in \{1,2,3\})$ *has a terminal segment which lies entirely in a neighbourhood of the vertical line segment* $\{k\} \times [3n+1, 3n+3]$ *and is isotopic to* S^{tm}_{r,j_0} *(if it is the automorphism* α_{j_0} *which is acting) or* U^{tm}_{r,j_0} *(if it is the inverse automorphism* $\alpha_{j_0}^{-1}$*).*

The claim asserts that given the word w_i and the curve diagram of the braid $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i)$, we can reconstruct which were the word replacements (i.e. the cells) that created each of the letters of w_i in a reduced diagram. In particular, the curve diagram of the full product $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_n)$ is nontrivial. Therefore, the claim implies proposition 2.1.

Part (a) of the claim is obvious. Part (b) is proved by induction on i. We assume inductively that the claim holds for the curve diagram of $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i)$. We now take one additional cell Δ_{i+1}. Without loss of generality we can assume that Δ_{i+1} is of the form $w_i = p'q_ip'' \stackrel{\Delta_{i+1}}{\longrightarrow} p'q_i'p''$, with $\Phi(\Delta) = \prod_{m=1}^{|q_i|} \beta_{|p'|+m,\lambda+m} \cdot \delta_{|p'|+1,|p'|+|q_i|} \cdot \epsilon_{|p'|+1}^{|q_i'|-|q_i|} \cdot \delta_{|p'|+1,|p'|+|q_i'|} \cdot \prod_{m=1}^{|q_i'|} \beta_{|p'|+m,\lambda'+m}$ as before (the only other possibility is that Δ_{i+1} might be the inverse of such a cell – in this case, a symmetric argument works). We consider whether the claim still holds for the curve diagram of $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i \cdot \Delta_{i+1})$. It certainly does in all columns corresponding to letters of w_{i+1} which are not in $\mathbf{bot}(\Delta_{i+1})$.

We shall say that a curve of a reduced curve diagram terminating at the particle in position (k,l) *ends vertically* if it has a terminal segment which intersects one of the horizontal lines $\mathbb{R}_+ \times \{l \pm 1\}$ but not the line $\mathbb{R}_+ \times \{l\}$. Similarly, a curve *ends horizontally* if a terminal segment intersects a vertical line adjacent to the kth one, but not the kth one itself. We remark that one and the same curve can end both vertically and horizontally.

The crucial observations now are the following: if a curve of a curve diagram ends horizontally at a particle (k,l), then after acting by the braid β_{k,j_0} the curve terminating at this particle ends vertically, and more precisely has a terminal segment isotopic to $S^{\mathrm{tm}}_{(l \bmod 3, j_0)}$. Similarly, if a curve of a curve diagram ends vertically at a particle (k,l), and if $k_0, k_1 \in \mathbb{N} \setminus \{0\}$ with $k_1 - k_0 \geqslant 2$ and $k_0 \leqslant k \leqslant k_1$, then after acting by the braid δ_{k_0,k_1} we have that all curves of the curve diagram terminating at a particle at position (k', l) with $k_0 \leqslant k' \leqslant k_1$ end horizontally.

Since our diagram is, by hypothesis, reduced, the cell Δ_{i+1} does not cancel any of the previous cells. In particular, there exists at least one position in $\mathbf{top}(\Delta_{i+1})$, say the $|p'|+l$th one ($l \in \{1, \ldots, |q_i|\}$), such that $\beta_{|p'|+l,\lambda+l}$ is *not* the inverse of the last pseudo-Anosov braid that acted on the $|p'|+l$th column during the particle dance $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i)$. Therefore the curve diagram of $\Phi(\Delta_1 \cdot \ldots \cdot \Delta_i) \cdot \beta_{|p'|+1,\lambda} \cdot \ldots \cdot \beta_{|p'|+|q_i|,\mu}$ has the following property: the curve terminating at a particle in the $|p'|+l$th column, say at the position $(|p'|+l, 3n+r)$, ends vertically, and more precisely is has an terminal segment isotopic to $S^{\mathrm{tm}}_{r,\lambda+l}$. (It is not true in general that *all* the columns $|p'|+1,\ldots,|p'|+|q'|$ of the curve diagram will contain terminal segments of stable or unstable foliations, but the $|p'|+l$th one will.) We now apply the braid $\delta_{|p'|+1,|p'|+|q_i|}$, and consider the curve diagram of $\Phi(\Delta_1\cdot\ldots\cdot\Delta_i)\cdot\prod\beta_{|p'|+m,\lambda+m}\cdot\delta_{|p'|+1,|p'|+|q_i|}$. Since $|q_i|\geqslant 2$, we can apply the above observation, and conclude that all curves terminating at particles in columns number $|p'|+1,\ldots,|p'|+|q_i|$ end horizontally. Next we consider the effect of acting by the braid $\epsilon_{|p'|+1}^{|q_i'|-|q_i|}$, which, we recall, moves particles in columns number $|p'|+1$ and $|p'|+2$ in a predominantly vertical direction. It is easy to check that in the curve diagram of $\Phi(\Delta_1\cdot\ldots\cdot\Delta_i)\cdot\prod\beta_{|p'|+m,\lambda+m}\cdot\delta_{|p'|+1,|p'|+|q_i|}\cdot\epsilon_{|p'|+1}^{|q_i'|-|q_i|}$, in every row, except possibly the bottom one, the curve terminating at the particle in the $|p'|+1$st or $|p'|+2$nd position ends vertically. Therefore, if we finally apply the braid $\delta_{|p'|+1,|p'|+|q_i'|}$ to this curve diagram, then the curves of the resulting curve diagram which terminate at particles in positions (k,l) with $|p'|+1\leqslant k\leqslant|p'|+|q_i'|$ and $l>1$ end horizontally. About the particles in the bottom row we cannot make such a strong statement, but at least we know that the curve ending in position $(k,1)$ with $|p'|+1\leqslant k\leqslant|p'|+|q_i'|$ does not have a terminal segment isotopic to $U_{(1,j)}^{\mathrm{tm}}$ for any $j\in\{1,\ldots,W\}$, for in order to do so it would have to intersect the predominantly horizontal curves in the higher rows. Therefore, if we finally act by the braid $\prod_{m=1}^{|q_i'|}\beta_{|p'|+m,\lambda'+m}$, we can indeed apply the above observation; we conclude that the curve of the curve diagram of $\Phi(\Delta_1\cdot\ldots\cdot\Delta_i\cdot\Delta_{i+1})$ which terminates at the particle in position $(|p'|+m,3n+r)$ (with $1\leqslant m\leqslant|q_i'|$, $n\in\mathbb{N}$, $r\in\{1,2,3\}$) has a terminal segment isotopic to $S_{(l\bmod 3,\lambda'+\mathrm{m})}^{\mathrm{tm}}$. This concludes the inductive step, and thus the proof of the claim and of proposition 2.1. $\square$

3. The braid group $\mathcal{B}$ is left-orderable

Proposition 3.2. *There exists a total order $<$ on the group $\mathcal{B}$ which is invariant under left multiplication.*

Proof The proof is very similar in spirit to the proof in [4] using curve diagrams that the classical braid groups (even those on infinitely many strings) are left-orderable. Therefore we shall not go through all the details of the proof. (A wealth of information about orderings of braid groups can be found in [2].)

Let $\beta\in\mathcal{B}$ be a nontrivial braid; we want to define whether $\beta>1$ or $\beta<1$. Our definition should have the property that the product of positive braids is again positive – once this condition is satisfied, we can extend our ordering to a left-invariant total order on $\mathcal{B}$ by the rule $\beta_1<\beta_2:\Longleftrightarrow\beta_2^{-1}\beta_1<1$.

We start by fixing, once and for all, an enumeration of the set of pairs (i,j) with $i,j\in\mathbb{N}\setminus\{0\}$ – this yields, in particular, an enumeration of the set of line segments from $(x,0)$ to the particles. Since the two curve diagrams D and $\beta(D)$ are not homotopic, we can choose (i,j) so that $D_{i,j}$ is the first line segment in our enumeration for which $D_{i,j}$ and $\beta(D_{i,j})$ are not homotopic. (Here $\beta(D_{i,j})$ denotes the reduced curve obtained from $D_{i,j}$ by the action of a homeomorphism representing β.)

Let us compare the two curves $D_{i,j}$ and $\beta(D_{i,j})$. After a further homotopy of β we can assume that $D_{i,j}$ and $\beta(D_{i,j})$ are *tight* with respect to each other, meaning that they only intersect transversely, and that any bigon enclosed by one arc of $D_{i,j}$ and one arc of $\beta(D_{i,j})$ contains at least one particle. Once the two curves are in such a tight position, we consider arbitrarily small initial segments of the two curves at the point $(x,0)$. If an initial segment of $\beta(D_{i,j})$ has a greater angle than $D_{i,j}$ with the ray $(x,\infty)\times\{0\}$, i.e. if the curve $\beta(D_{i,j})$ "goes more to the left than the curve $D_{i,j}$", then we say $\beta > 1$; otherwise we say $\beta < 1$. The proof that this yields a well-defined, and left-invariant ordering of $\mathcal{B}$ is completely analogue to the proof in the classical case [4]. $\square$

Remark 3.1. In order to prove that diagram groups are bi-orderable, it would suffice to prove that they all embed in the *pure* analogue of the braid group $\mathcal{B}$, and that this latter group is bi-orderable. The second step should be relatively easy using techniques similar to [11], but the first step may be quite difficult.

As an alternative approach, it would be interesting to know if diagram groups embed in the group of PL homeomorphisms of a closed interval of the real line, which is well-known to be bi-orderable. One might send a cell $w_i \xrightarrow{\Delta_i} w_{i+1}$ to a PL homeomorphism $[0,|w_i|]\to[0,|w_{i+1}|]$.

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