An elementary approach to quasi-isometries of tree$\times\mathbb{R}^n$

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Abstract. We prove by elementary means a regularity theorem for quasi-isometries of $T\times\mathbb{R}^n$ (where T denotes an infinite tree), and of many other metric spaces with similar combinatorial properties, e.g. Cayley graphs of Baumslag-Solitar groups. For quasi-isometries of $T\times\mathbb{R}^n$ it states that the image of $\{x\}\times\mathbb{R}^n$ ($x\in T$) is uniformly close to $\{y\}\times\mathbb{R}^n$ for some $y\in T$, and there is a well-defined surjection $QI(T\times\mathbb{R}^n)\to QI(T)$. Even stronger, the image of a quasi-isometric embedding of $\mathbb{R}^{n+1}$ in $T\times\mathbb{R}^n$ is close to (a geodesic in T)$\times\mathbb{R}^n$.

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1. Motivation and statement of results

What do quasi-isometries of the Cayley graph of $F_2\times\mathbb{Z}^n$ look like? As a special case of a powerful and difficult theorem in (??) we know the answer; it is, roughly speaking: there are only the obvious ones! The aim of the present paper is to give an elementary proof of this and in fact of the stronger result that the image of a quasi-isometric embedding of $\mathbb{R}^{n+1}$ in $F_2\times\mathbb{R}^n$ is uniformly close to (a geodesic in F_2)$\times\mathbb{R}^n$. This is done in section 2. Moreover, due to the mostly topological (not geometric) nature of the proof, the hypotheses can be weakened considerably, and we shall adapt our techniques to other situations, different from the ones considered in (??). For instance, we shall give a simple proof of a key result of (??) on quasi-isometries of Baumslag-Solitar groups, and, more generally, of results of (??) and (??) on graphs of $\mathbb{Z}^n$s. Also, our results apply to general spaces of the form tree $\times\,\mathbb{R}^n$, where the hypotheses on the metrics carried by the tree and by $\mathbb{R}^n$ are very weak –e.g. we may take tree $\times\,\mathbb{H}^n$. Moreover, instead of a quasi-isometry we can use any uniform embedding, and the conclusion still holds true. This is the content of section 3. In a final section, we give some surprising examples, which illustrate the difficulty in trying to apply our techniques to other product spaces like $F_2\times F_2$, or to semidirect products $F_2\rtimes\mathbb{Z}$.

To fix notation, we recall that a (λ, C)-*quasi-isometric embedding*, for $\lambda > 1, C > 0$, is a function $f\colon X \to Y$ between metric spaces such that $\frac{1}{\lambda} d_X(x,\tilde{x}) - C \leqslant d_Y(f(x), f(\tilde{x})) \leqslant \lambda d_X(x,\tilde{x}) + C$ for all $x, \tilde{x} \in X$. We shall often talk about a "quasi-isometric embedding", without specifying the so-called *quasi-isometry constants* λ and C of f. A *quasi-isometry* is by definition a quasi-isometric embedding which is quasi-surjective (meaning that there exists a $D > 0$ such that the D-neighbourhood of $im(f)$ in Y equals all of Y.) A quasi-isometry f has a *quasi-inverse*, i.e. a quasi-isometric embedding $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are uniformly close to the identity on Y and X respectively; moreover, the quasi-isometry constants of g can be bounded in terms of D, λ, and C. On the monoid of all quasi-isometries of X into itself, one can define an equivalence relation by identifying two quasi-isometries which are uniformly close to each other; the quotient naturally carries a group structure, this is the *quasi-isometry group* $QI(X)$. For details and background see (??; ??; ??).

Throughout the paper we denote by T (or T') an infinite simplicial tree in which the length of all edges is finite and bounded away from zero and each vertex has finite valency (note that the valency is not necessarily uniformly bounded). Recall that a metric space (X, d) is *uniformly contractible* if there is a function $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that any continuous map of a finite simplicial complex (not necessarily connected) to X whose image is contained in an r-ball is contractible in an $M(r)$-ball. For example, any contractible space with a co-compact isometry group is uniformly contractible.

2. Tree$\times\mathbb{E}^n$

2.1. Statements

We shall consider spaces of the form $T \times \mathbb{E}^n$, where $\mathbb{E}^n$ is the Euclidean space of dimension n and $\times$ denotes the product metric (with the distance between points in the product space given by the Pythagorean formula). For instance, the Cayley graphs of $F_m \times \mathbb{Z}$, where F_m is a free group on m generators, are quasi-isometric to such spaces (with $\mathbb{R}$ as the second factor).

Lemma *If $\varphi\colon \mathbb{E}^{n+1} \to T \times \mathbb{E}^n$ is a quasi-isometric embedding, then the image of φ is uniformly close to a "hyperplane in $T \times \mathbb{E}^n$", i.e. to $L \times \mathbb{E}^n$, where L is a bi-infinite geodesic in T.*

THEOREM 2.1. *Suppose $f\colon T \times \mathbb{E}^n \to T' \times \mathbb{E}^n$ is a quasi-isometric embedding, and all vertices of T have valency at least 3. Then f "preserves the $\mathbb{E}^n$-direction". More precisely, there is a constant $D = D(f) > 0$*

with the property that for any vertex $x' \in T$ there is a vertex $y' \in T'$ such that $f(\{x'\} \times \mathbb{E}^n)$ and $\{y'\} \times \mathbb{E}^n$ have Hausdorff-distance at most D.

Remark In fact, if f is a (λ, C)-quasi-isometry, then one can bound D in terms of λ and C. The proof of this fact is not hard, but tedious; one simply has to give bounds on the relevant constants of quasi-isometry in every step of our proof.

COROLLARY 2.2. *Suppose each vertex of T has valency at least 3, and each edge of T has length 1. Then there is a well-defined and natural surjective homomorphism θ: $QI(T \times \mathbb{E}^n) \to QI(T)$. Thus $QI(T \times \mathbb{E}^n)$ is a semi-direct product $QI(T) \ltimes \ker(\theta)$, where $\ker(\theta)$ consists of all quasi-isometries of $T \times \mathbb{E}^n$ which fix the T-coordinate.*

Remarks

1. This result is the best possible: there do exist quasi-isometries of $T \times \mathbb{R}$ which are not close to ones of the form $f' \times f''$, where f' and f'' are quasi-isometries of T and $\mathbb{R}$. For instance, for any fixed $x'_0 \in T$ consider the "shear" $(x', x'') \mapsto (x', x'' + d_T(x', x'_0))$.

2. It is not difficult to find a characterization of the elements of $\ker(\theta)$. Let $(\varphi_t)_{t \in T}$ be a family of mappings $\mathbb{E}^n \to \mathbb{E}^n$. For $(t, x) \in T \times \mathbb{E}^n$, let us define $\varphi(t, x) = (t, \varphi_t(x))$. Then φ is a quasi-isometry of $T \times \mathbb{E}^n$ (thus belonging to $\ker \theta$) if and only if:

   (i) There exist $\lambda \geq 1$ and $C \geq 0$ such that for all $t \in T$, φ_t is a (λ, C)-quasi-isometry.

   (ii) There exist $\mu > 0$ and $D \geq 0$ such that for all $t, t' \in T$ and $x \in \mathbb{E}^n$, $d_{\mathbb{E}^n}(\varphi_t(x), \varphi_{t'}(x)) \leq \mu d_T(t, t') + D$.

Proof of the Corollary . Since each point of T is at bounded distance from a vertex, it follows from theorem 2.1 that a (λ, C)-quasi-isometry f of $T \times \mathbb{E}^n$ induces a well-defined mapping f': $T \to T$; we have to prove f' is a quasi-isometry. Since the projection of $T \times \mathbb{E}^n \to T$ is distance-decreasing, we have $d(f'(t), f'(\tilde{t})) \leqslant \lambda d(t, \tilde{t}) + C$ for all $x, \tilde{x} \in T$. If the quasi-inverse g of f is a (λ', C')-quasi-isometry, then the projection g': $T \to T$ is, by the same arguments, a mapping $T \to T$ with $d(g'(x), g'(\tilde{x})) \leqslant \lambda' d(t, \tilde{t}) + C''$ such that $f' \circ g'$ and $g' \circ f'$ are both close to the identity of T. It follows that f' and g' are quasi-isometries.

2.2. PROOFS

We start by presenting a lemma whose proof is a prototype for the sort of argument we shall need for theorem 2.1. This very elegant proof appears to be due to Brian Bowditch (??). We shall give an alternative proof, which allows broader generalisations, later on; thus parts of the present proof are only included for their beauty, and are not strictly necessary for what follows. The result was certainly known before - see (??) Lemma 8.2, (??) Cor. 5.3.

LEMMA 2.3.
All quasi-isometric embeddings $\mathbb{E}^n \to \mathbb{E}^n$ are quasi-surjective.

Proof. Suppose $f\colon \mathbb{E}^n \to \mathbb{E}^n$ is a (λ, C)-quasi-isometric embedding. The first step is to observe that there is a *continuous* mapping $f_1\colon \mathbb{E}^n \to \mathbb{E}^n$ which is uniformly close to f, i.e. such that $d_{\mathbb{E}^n}(f(x), f_1(x))$ is globally bounded.

Here is a sketch proof of this fact, known as the "connect-the-dots argument". We fix a CW structure of the domain $\mathbb{E}^n$, where the size of all cells is globally bounded by a constant M. Now we build up f_1 by induction on the dimension of the cells. For every vertex v let $f_1(v) = f(v)$. We send each edge e of the CW structure to a geodesic segment between $f_1(\imath(e))$ and $f_1(\tau(e))$. Note that f_1, as defined so far (on the 1-skeleton) is uniformly close to f, because for any point x on e, both $f_1(x)$ and $f(x)$ have distance at most $\lambda M + C$ from $f(\imath(e))$. If a continuous map f_1 on the boundary of a certain i-cell is already given, and if the image of this $(i-1)$-sphere by f lies in some ball $B_\epsilon(x)$ in the target space, then we can find a *continuous* mapping of the i-cell which extends the mapping of the boundary, and whose image lies in the same ball. The fact that the size of the cells is uniformly bounded and that f verifies the right inequality ensures that $d_{\mathbb{E}^n}(f(x), f_1(x))$ is globally bounded. This completes the construction of f_1.

Thus by replacing f by f_1 and possibly increasing C we obtain a continuous quasi-isometric embedding close to f. Our aim is now to prove the stronger result that $f_1\colon \mathbb{E}^n \to \mathbb{E}^n$ is actually *surjective.*

The second step is to extend f_1 to infinity: let us consider the one-point compactification of $\mathbb{E}^n$, such that the neighbourhoods of ∞ are the complements of the compact subsets of $\mathbb{E}^n$. Let $\overline{f_1}$ be such that $\overline{f_1}(x) = f_1(x)$ if $x \in \mathbb{E}^n$ and $\overline{f_1}(\infty) = \infty$. In order to prove that $\overline{f_1}$ is continuous, it suffices to show that f_1 is a proper mapping, that is, that the preimages of any compact (i.e. closed and bounded) $A \subset \mathbb{E}^n$ is again compact. But $f^{-1}(A)$ is closed since f_1 is continuous, and bounded since f_1 is a quasi-isometric embedding (using the left inequality). Thus f_1 is proper, and extending f_1 by sending ∞ to ∞ yields a *continuous* mapping $\overline{f}_1\colon \mathbb{E}^n \cup \infty \to \mathbb{E}^n \cup \infty$.

There is a homeomorphism $\delta\colon S^n \to \mathbb{E}^n \cup \infty$ identifying $\mathbb{E}^n \cup \infty$ with the n-sphere; we note that simply by scaling we can choose δ in such a way that for any pair $x_a,\ x_b$ of antipodal points of S^n we have $d_{\mathbb{E}^n}(\delta(x_a), \delta(x_b)) > \lambda \cdot C$.

The third step is to use the Borsuk-Ulam theorem: let's assume, for a contradiction, that $\overline{f}_1$ is not surjective, i.e. that the image of $\overline{f}_1$ is in fact contained in a subset of $\mathbb{E}^n \cup \infty$ which is homeomorphic to $\mathbb{E}^n$. Then by the Borsuk-Ulam theorem (??) there exists a pair of *antipodal* points x_a and x_b in S^n such that $\overline{f}_1(\delta(x_a)) = \overline{f}_1(\delta(x_b))$. This is absurd: the (λ, C)-quasi-isometry $f_1\colon \mathbb{E}^n \to \mathbb{E}^n$ cannot identify two points that are more than $\lambda \cdot C$ away from each other, nor does it send any point of $\mathbb{E}^n$ to infinity. Therefore f_1 is surjective, and so f is quasi-surjective.

Remarks (1) We have proved that there exists a $K > 0$ such that every point of $\mathbb{E}^n$ has distance at most K from $im(f)$. We remark that K can be bounded in terms of λ and C.

(2) When considering the hyperbolic space $\mathbb{H}^n$ instead of $\mathbb{E}^n$, there is a well-known alternative proof using the sphere at infinity.

Proof. **of theorem 2.1** We fix a CW structure of $T \times \mathbb{E}^n$ in which the size of the cells is globally bounded, and each compact subset intersects only finitely many cells. As in step one of the proof of lemma 2.3, since $T \times \mathbb{E}^n$ is uniformly contractible, we can replace the quasi-isometric embedding f of $T \times \mathbb{E}^n$ into itself by a continuous and proper one. After a further bounded homotopy we can assume that f is *transverse*. To define what that means and justify it we need some general comments. (We shall use a non-standard definition of transversality, which is convenient for our purposes.)

removed "(see lemma 3.1)" - c'est plus tard!?!

Suppose $f\colon X \to Y$ is a continuous proper mapping between simplicial or CW complexes of dimension $(n+1)$, both of which have the property that each cell is in the closure of an $(n+1)$- (i.e. maximal)-dimensional one, and with only finitely many cells intersecting any compact subset. We call a point in X or Y *generic* if it is in the interior of an $(n+1)$-dimensional cell. We say f is *transverse* if for any open $(n+1)$-dimensional cell $\Delta \in Y$ we have that $f^{-1}(\mathrm{int}(\Delta))$ has only finitely many path components, each of which lies in the interior of some $(n+1)$-cell of X, and if the restriction of f to any such path component is a homeomorphism onto Δ. It is a basic fact from PL topology that any continuous proper map f is homotopic to a transverse one, where the homotopy moves points in Y only within their $(n+1)$-cells.

rajouté "open" – B

Now a point y in the interior of an $(n+1)$-cell $\Delta \subset Y$ has only finitely many preimages (all of them generic in X) which can be counted either *geometrically* or *modulo 2*. The geometric number is denoted $\mid f^{-1}(y) \mid$.

We recall that if X and Y are compact manifolds, then the modulo 2 count yields the *degree* modulo 2 of f. (We need the hypotheses on the local finiteness of the CW decomposition and that f be proper in order to get only finitely many cell preimages.)

LEMMA 2.4. *Let* $\varphi\colon \mathbb{E}^{n+1} \to T \times \mathbb{E}^n$ *be a continuous quasi-isometric embedding. Then its image* $\varphi(\mathbb{E}^{n+1})$ *contains a "hyperplane"* $L \times \mathbb{E}^n$*, where* L *is a bi-infinite geodesic in* T*.*

Proof. By the same methods as before, we can assume that φ is transverse. Let Y be the union of the closed $(n+1)$-cells intersecting $\varphi(\mathbb{E}^{n+1})$. Then the mapping φ has a quasi-inverse $\psi\colon Y \to \mathbb{E}^{n+1}$, which can also be made continuous and transverse. By definition of a quasi-inverse we have that $\psi \circ \varphi\colon \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ is uniformly close to the identity map, so there is a homotopy between $\psi \circ \varphi$ and $id_{\mathbb{E}^{n+1}}$ which moves every point of $\mathbb{E}^{n+1}$ only by a globally bounded distance, namely $H(x,t) = tx + (1-t)\psi \circ \varphi(x)$. Moreover, for all $t \in [0,1]$, the mapping $t \mapsto H(t,x)$ is uniformly close to the identity map, so it is proper. So by the same argument as in the proof of lemma 2.3, $\psi \circ \varphi$ and its homotopy with the identity map can all be extended to a family of mappings $\mathbb{E}^{n+1} \cup \infty \to \mathbb{E}^{n+1} \cup \infty$. Since the degree of mappings between compact manifolds is invariant under homotopy, and the identity map has degree 1, it follows that $\overline{\psi \circ \varphi}\colon \mathbb{E}^{n+1} \cup \infty \to \mathbb{E}^{n+1} \cup \infty$ has degree 1.

We now claim that there exists a point $y_0 \in T \times \mathbb{E}^n$ in the interior of an $(n+1)$-cell, such that $\varphi^{-1}(y_0) \subset \mathbb{E}^{n+1}$ has an odd number of points. To see this, it suffices to note that for a generic point $x \in \mathbb{E}^{n+1}$ we have that $\psi^{-1}(x) \subset T \times \mathbb{E}^n$ has a finite number of points; all of them are again generic and we can consider their preimages under φ. Since $(\psi \circ \varphi)^{-1}(x) \in \mathbb{E}^{n+1}$ has an odd number of points, and since

$$\mid (\psi \circ \varphi)^{-1}(x) \mid \; = \sum_{y \in \psi^{-1}(x)} \mid \varphi^{-1}(y) \mid ,$$

there exists necessarily at least one point $y_0 \in \psi^{-1}(x)$ such that $\varphi^{-1}(y_0) \subset \mathbb{E}^{n+1}$ has an odd number of points. This proves the claim.

Next, we define a projection mapping $proj_{y_0}\colon T \times \mathbb{E}^n \to \mathbb{R} \times \mathbb{E}^n$ that "folds the tree into a line, pivoting around the point y_0" (see figure 1). More precisely, for every $y \in T \times \mathbb{E}^n$ let $y' \in T$ and $y'' \in \mathbb{E}^n$ denote the two coordinates of y, so that for instance $y_0 = (y_0', y_0'')$. Then we define a projection $proj_{y_0}'\colon T \to \mathbb{R}$ which sends y_0' to 0, and the two path components of $T - x_0'$ onto $\mathbb{R}_+$ and $\mathbb{R}_-$, in such a way that for every $y' \in T$ we have $\mid proj'(y') \mid \; = d_T(y', y_0')$. Finally, we define $proj_{y_0} = proj_{y_0}' \times id_{\mathbb{E}^n}$.

units ¡0.47000cm,0.47000cm¿ 1pt (.) (.) 1pt (.) axes ratio 0.080:0.080 360 degrees from 10.717 21.590 c
[lB] at 9.842 21.114 y_{-1} [lB] at 4.445 23.019 y_0 [lB] at 4.128 21.749 e_{-1} [lB]
at 1.429 22.701 e_0 [lB] at 3.810 20.955 $\widetilde{y}_1$ [lB] at 24.765 22.066 e_1^2 [lB] at 6.7
20.45 e_1^3 [lB] at 6.668 22.225 $e_1 = e_1^1$ [lB] at 3.65 19.7 0pt corners at 0.133 25.267
and 27.965 19.412

Figure 1. The projection $proj_{y_0}: T \times \mathbb{E}^n \to \mathbb{R} \times \mathbb{E}^n$ (here $n = 1$, and all edges adjacent to e_0 have the same length)

We now consider the composition $proj_{y_0} \circ \varphi: \mathbb{E}^{n+1} \to \mathbb{R} \times \mathbb{E}^n$. Since both φ and $proj_{y_0}$ are proper, by the same arguments as before, it can be extended to a mapping between the compactified spaces $\overline{proj_{y_0} \circ \varphi}: \mathbb{E}^{n+1} \cup \infty \to \mathbb{R} \times \mathbb{E}^n \cup \infty$. Moreover, if $e_0 \subset T$ denotes the edge which contains y_0', then we observe that the restriction of $proj_{y_0}$ to $e_0 \times \mathbb{E}^n$ is a homeomorphism (in fact an isometry). Thus, by counting preimages of the point $proj_{y_0}(y_0)$, we find that the map $\overline{proj_{y_0} \circ \varphi}$ has odd degree. In particular, it is surjective, which implies in turn that the whole "band" $e_0 \times \mathbb{E}^n \subset T \times \mathbb{E}^n$ is in the image of φ.

Let us now consider the edges e_1^i of T adjacent to one of the endpoints of e_0 (see Figure 1). The images of the bands $e_1^i \times \mathbb{E}^n$ under the projection $proj_{y_0}$ to $\mathbb{R} \times \mathbb{E}^n$ all contain a band of width $\min_i\{\mathrm{length}(e_1^i)\}$. Since $\overline{proj_{y_0} \circ \varphi}: \mathbb{E}^{n+1} \to \mathbb{R} \times \mathbb{E}^n$ has odd degree, we have for any generic point $\widetilde{y}_1$ in this image band that $|(proj_{y_0} \circ \varphi)^{-1}(\widetilde{y}_1)|$ is odd. It follows that at least one of the points of $proj_{y_0}^{-1}(\widetilde{y}_1) \subset T \times \mathbb{E}^n$ has an odd number of preimages under φ. Let's call this point y_1, and let's denote the corresponding edge among the edges e_1^i by e_1. Now we can use a projection $proj_{y_1}$ pivoting around y_1 to prove, by the same arguments as before, that the whole band $e_1 \times \mathbb{E}^n$ is in the image of φ, and covered by φ with degree 1.

Finally, with the help of the projection $proj_{y_1}$, we find a point y_2 in a band $e_2 \times \mathbb{E}^n$, and so on. It follows inductively that the image of φ contains a whole hyperplane $L \times \mathbb{E}^n$, where L is a geodesic $L := \bigcup_{i \in \mathbb{Z}} e_i$.

Remark Lemma 2.4 enables us to give another proof of lemma 2.3, without using the Borsuk-Ulam theorem:

COROLLARY 2.5. *All quasi-isometric embeddings $\mathbb{E}^n \to \mathbb{E}^n$ are quasi-surjective.*

Proof. It suffices to apply lemma 2.4 in the special case when T is a regular tree of valency 2, i.e. is isometric to $\mathbb{R}$. In this case, we do not even need the procedure of successive projections; the observation that φ has odd degree suffices.

LEMMA 2.6. *Let $\varphi: \mathbb{E}^{n+1} \to T \times \mathbb{E}^n$ be a continuous quasi-isometric embedding. Then there exists a constant $D' = D'(\lambda, C)$ such that the* *image $\varphi(\mathbb{E}^{n+1})$ lies in the D'-neighbourhood of a hyperplane $L \times \mathbb{E}^n$, where $L \subset T$ is a bi-infinite geodesic.*

Proof. We recall that there is a continuous (λ', C')-quasi-isometric embedding $\psi\colon \varphi(\mathbb{E}^{n+1}) \to \mathbb{E}^{n+1}$ (for some $\lambda', C' \in \mathbb{R}_+$ which can be bounded in terms of λ and C), namely the quasi-inverse of φ. Moreover, by the previous lemma, the image of φ contains a hyperplane (which is isometric to $\mathbb{R} \times \mathbb{E}^n$), we only have to prove that it can't contain much more than that. By lemma 2.3 or lemma 2.5, the restriction of ψ to the hyperplane is already surjective; it follows that $\varphi(\mathbb{E}^{n+1}) \subseteq T \times \mathbb{E}^n$ can't contain any point whose distance to the hyperplane is greater than $\lambda' \cdot C'$, for ψ cannot identify points which are more than $\lambda' \cdot C'$ away from each other.

We can now finish the proof of theorem 2.1, using an argument which is similar to the one of section 7.2 of (??): if $x' \in T$, then we want to study $f(\{x'\} \times \mathbb{E}^n)$. We know from lemma 2.6 that f induces an injective function F from the set of bi-infinite geodesics of T to that of T'; F can be defined by the fact that f sends any hyperplane $L \times \mathbb{E}^n$ near a hyperplane $F(L) \times \mathbb{E}^n$. Now consider a tripod centered at $x' \in T$, i.e. three disjoint geodesic rays T_1, T_2, T_3 emenating from x'. Any two of these rays can be combined to form a bi-infinite geodesics $L_{1,2}, L_{1,3}$ or $L_{2,3}$. We note that $F(L_{1,2})$ and $F(L_{1,3})$ are not identical (because all points in $T_3 \times \mathbb{E}^n$ whose distance to $L_{1,2} \times \mathbb{E}^n$ is larger than $\lambda \cdot (C + 2D')$ cannot be sent to $F(L_{1,2}) \times \mathbb{E}^n$). Moreover, each geodesic $L_{i,j}$ is contained in the union of the other two, and the same goes for their images $f(L_{i,j} \times \mathbb{E}^n)$; each such image contains a unique hyperplane $F(L_{i,j}) \times \mathbb{E}^n$, so each of *them* must be contained in the union of the other two. We deduce that $F(L_{1,2}) \cup F(L_{1,3}) \cup F(L_{2,3})$ is a tripod in T', centered at some point y'. We now have that $f(\{x'\} \times \mathbb{E}^n) = f((L_{1,2} \cap L_{1,3} \cap L_{2,3}) \times \mathbb{E}^n) \subseteq (N(L'_{1,2}) \cap N(L'_{1,3}) \cap N(L'_{2,3})) \times \mathbb{E}^n$, where N denotes the $\lambda' \cdot C'$-neighbourhood. Since $N(L'_{1,2}) \cap N(L'_{1,3}) \cap N(L'_{2,3})$ is a finite neighbourhood of the point y', we have completed the proof of theorem 2.1.

3. Generalisations

3.1. Other metrics on $\mathbb{R}^n$

In all the previous section, the metric on $\mathbb{R}^n$ was the Euclidean metric. We can observe that most of the arguments in the previous section still work if we replace the Euclidean metric on the topological space $\mathbb{R}^n$ by any "reasonable" metric.

Let's, for the moment, call a metric on $\mathbb{R}^n$ *tame* if it is proper, uniformly contractible, and admits a CW-decomposition of $\mathbb{R}^n$ whose cells have uniformly bounded diameter, and with only finitely many cells intersecting any compact subset of $\mathbb{R}^n$. For instance, $\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^n$, equipped with a tame metric.

Enlevé "We need the following lemmas:" – B.

LEMMA 3.1. *Let d, d' be tame metrics on $\mathbb{R}^n$. Then any quasi-isometric embedding $(\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ is uniformly close to a continuous mapping.*

Proof. This is a simple generalisation of the "connect the dots" argument presented in the proof of lemma 2.3

added this "proof" – B.

LEMMA 3.2. *Let d, d' be tame metrics on $\mathbb{R}^n$. Let $f\colon (\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ be a continuous mapping, and suppose there exists a constant $K > 0$ such that $d'(f(x), x) \leqslant K$ for all $x \in \mathbb{R}^n$. Then the continuous extension of f to the one-point compactification of $\mathbb{R}^n$ has degree one.*

Proof. The first step is to construct a homotopy $G\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ between f and the identity map (i.e. with $G(x,0) = f(x)$ and $G(x,1) = x$) such that $sup_{x\in\mathbb{R}^n, t\in[0,1]} d(G(x,t), x) < \infty$.

Sinc d is tame, there is a CW structure on $\mathbb{R}^n$ such that the diameter of all cells for d is uniformly bounded by a constant M. It induces a CW structure on $\mathbb{R}^n \times [0,1]$, still with cells of uniformly bounded diameter.

We define G on the cells in $\mathbb{R}^n \times (0,1)$ by induction on the dimension, starting with $i = 1$: suppose G is already defined on the $(i-1)$-skeleton of $\mathbb{R}^n \times [0,1]$, in such a way that the images under G of the boundaries of all i-cells have globally bounded diameter. Then we can extend G over the i-cells, and since the metric d' on the target space $\mathbb{R}^n$ is uniformly contractible, we can do this in such a way that the diameter of the images of the i-cells is again globally bounded.

Since f is uniformly close to the identity map, it is proper, and can be continuously extended to a mapping $\overline{f}\colon \mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty$. As in the proof of lemma 2.4, the homotopy G between f and the identity map on $\mathbb{R}^n$ can be extended to $\mathbb{R}^n \cup \infty$. Since the degree of mappings between compact manifolds is invariant by homotopy, $\overline{f}$ has degree 1.

PROPOSITION 3.3.

(a) The analogue of theorem 2.1 holds for the metric space $T \times \mathbb{R}^n$, where $\mathbb{R}^n$ carries any tame metric.

(b) Let d, d' be two tame metrics on $\mathbb{R}^n$. Then all quasi-isometric embeddings $(\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ are quasi-surjective.

3.2. Graphs of $\mathbb{Z}^n$s

We shall now show that our previous results also hold for a larger class of spaces, namely the graphs of groups in which all vertex groups and edge groups are $\mathbb{Z}^n$.

This class of groups includes for example all the Baumslag-Solitar groups, which have been studied in (??) and (??), and also finitely presented non polycyclic abelian-by-cyclic groups, which are special cases of HNN-extensions of $\mathbb{Z}^n$, and have been treated in (??). Using powerful coarse separation results in (??), results similar to ours (and much more general results) have been proven in (??) and (??), in order to obtain quasi-isometric rigidity results for that class of groups and other, larger classes. It seems likely that our techniques can be applied to more general classes of graphs of groups than graphs of $\mathbb{Z}^n$s. We refer the reader to (??) for references concerning graphs of groups.

For our purposes, we don't need a detailed description of the Cayley graphs of such graphs of groups. All we need to know about them are the following well-known facts.

There is a metric space X quasi-isometric to the Cayley graph of the graph of groups and homeomorphic to $T \times \mathbb{R}^n$, where T denotes the Bass-Serre tree of the graph of groups. More precisely, X can be constructed by taking a copy of $\mathbb{R}^n$ for each vertex v of T, a copy of $\mathbb{R}^n \times [0,1]$ for each edge e of T, and if e connects the vertices $\tau(e)$ and $\imath(e)$, then we attach the boundaries $\mathbb{R}^n \times \{0\}$ respectively $\mathbb{R}^n \times \{1\}$ to the $\mathbb{R}^n$s lying over the vertices $\tau(e)$ and $\imath(e)$ respectively, by an (affine) linear transformation. Only a finite number of different such glueing maps occur in the whole space X, namely the natural extensions to $\mathbb{R}^n$ of the inclusion maps from edge- to vertex groups that appeared in the graph of groups. In particular, there is a constant $\lambda > 0$ such that all glueing maps are bilipschitz, dilating distances only by factors between $1/\lambda$ and λ.

If every $\mathbb{R}^n \times [0,1]$ is fibred by segments $\{x\} \times [0,1]$, then after the glueing we obtain a trivial fibration of X by copies of T. Choosing any point of X as the basepoint, we get a (piecewise linear) homeomorphism $\zeta\colon \mathbb{R}^n \times T \to X$ which identifies each $\mathbb{R}^n \times e$, where e is an edge of T, with a copy of $\mathbb{R}^n \times [0,1]$ in X, and in particular $\mathbb{R}^n \times \{0\}$ by the identity map with the horizontal plane through the base point. In particular, $d_X(\zeta(x,t), \zeta(x,t')) = d_T(t,t')$ for all $t, t' \in T$ and $x \in \mathbb{R}^n$. We stress that the map ζ is not canonical, but depends on the choice of basepoint for H.

Slightly different descriptions of metric spaces associated to such graphs of groups can be found in (??), (??) (for Baumslag-Solitar groups and graphs of $\mathbb{Z}$s), (??) (for finitely presented abelian-by-cyclic groups) and (??) (for graphs of $\mathbb{Z}^n$s).

We define a hyperplane H in X to be a subset homeomorphic to $\mathbb{R}^{n+1}$ corresponding to (a geodesic in T) $\times \mathbb{R}^n$. Note that hyperplanes are in general not quasi-convex in X. A hyperplane H consists of a $\mathbb{Z}$-family of "horizontal bands" $\mathbb{R}^n \times [0,1]$, where $\mathbb{R}^n$ is equipped with the Euclidean metric, and the boundary $\mathbb{R}^n \times \{1\}$ of the ith copy is identified with the boundary $\mathbb{R}^n \times \{1\}$ of the $(i+1)$st by a bilipschitz (and in fact affine linear) mapping with Lipschitz constant λ.

In the following lemma we shall, by abuse of notation, use the identification of H with $\mathbb{R}^n \times \mathbb{R}$ by ζ without explicitly writing the homeomorphism. Thus the Euclidean metric on $\mathbb{R}^n \times \mathbb{R}$ gives rise to a metric d_{Eucl} on H.

We remark that $\mathbb{R}^n \times \{t\}$ in H is in general not quasi-convex, and we shall denote by d_H the shortest path metric in H. Note that H, equipped with the metric d_H, is a geodesic metric space. We note as well that $d_H((x,t),(x',t')) \geqslant d_T(t,t')$ for all $(x,t),(x',t') \in H$.

LEMMA 3.4. *(a) There exists a mapping $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \rho(t) = \infty$ and and for all $h_1, h_2 \in H$ we have:*

$$\rho(d_H(h_1,h_2)) \leqslant d_X(h_1,h_2) \leqslant d_H(h_1,h_2).$$

(b) For every $K > 0$ there exist functions $\alpha_+, \alpha_-\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \alpha_\pm(t) = \infty$ and for all $h_1, h_2 \in \mathbb{R}^n \times [-K,K]$ we have

$$\alpha_-(d_{Eucl}(h_1,h_2)) \leqslant d_H(h_1,h_2) \leqslant \alpha_+(d_{Eucl}(h_1,h_2)).$$

Moreover, $\alpha_\pm$ can be chosen independently of the choice of basepoint in H.

Proof. (a) Let x, y be two points of H such that $d_X(x,y) = D$. Clearly $d_X(x,y) \leqslant d_H(x,y)$. Let γ be a geodesic in X between x and y. Considering the vertical projection of γ on H, we obtain that $d_H(x,y) \leq \lambda^D D$.

(b) We observe that there are such functions $\alpha'_\pm$ with $\alpha'_-(d_{Eucl}((x,t),(0,0))) \leqslant d_H((x,t),(0,0)) \leqslant \alpha'_+(d_{Eucl}((x,t),(0,0)))$. Moreover, we have by construction $d_H((x,t),(x',t')) = d_H((x+v,t),(x'+v,t'))$ for all $v \in \mathbb{R}^n$, and $d_H((x,t),(x',t'))$ and $d_H((x,t+u),(x',t'+u))$ differ only by a factor between $1/\lambda^{|u|}$ and $\lambda^{|u|}$.

LEMMA 3.5. *Let H be a hyperplane of X. Then (H, d_X) is uniformly contractible.*

Proof. Thanks to lemma 3.4(a), it suffices to prove that (H, d_H) is uniformly contractible.

We note that by lemma 3.4(b) the d_H-ball of radius $R > 0$ around $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ is contained in the "cylinder" $B(r) \times [-R, R]$, where $B(r)$ denotes of Euclidean ball of some sufficiently large radius $r = r(R)$ around $0 \in \mathbb{R}^n$. This cylinder is contractible, and in turn by lemma 3.4(b) contained in some sufficiently large d_H-ball. Since all these estimates were independent of the choice of the basepoint in H, we have (a).

LEMMA 3.6. *Let H and H' be two hyperplanes in X (both equipped with the d_X-metric). Then any quasi-isometric embedding $f\colon H \to H'$ is quasi-surjective.*

Proof. It is easy to see that (H, d_X) and (H', d_X) satisfy the hypotheses of lemma 3.3 (a).

LEMMA 3.7. *Let (H, d_X) be a hyperplane of X and $\varphi\colon H \to X$ a continuous quasi-isometric embedding. Then its image contains a hyperplane of X.*

Proof. The proof of lemma 2.4 goes through virtually unchanged. It is easy to find a CW structure of X in which the diameter of all cells is uniformly bounded, and every compact subset intersects only finitely many cells. By the same methods are before, we can assume that φ is continuous and transverse, and that its quasi-inverse $\psi\colon \varphi(H) \to H$ is continuous and transverse too. Then $\psi \circ \varphi\colon H \to H$ is a continuous map which is uniformly close to the identity map. H is homeomorphic to $\mathbb{R}^{n+1}$, the metric d_X of H is proper and uniformly contractible, and after applying lemma 3.4(a) we see that the conclusion of lemma 3.2 holds for H. Thus, if $\overline{\psi \circ \varphi}$ is the continuous extension of $\psi \circ \varphi$ to $H \cup \infty$, then $\overline{\psi \circ \varphi}$ has degree 1. Finally, we have to find a hyperplane in the image of φ "band by band", using appropriate projection maps $proj_{y_i}\colon X \overset{\zeta^{-1}}{\hookrightarrow} \mathbb{R}^n \times T \to \mathbb{R}^n \times \mathbb{R}$.

In fact, due to lemma 3.6, we even have (generalising lemma 2.6) that the image of a hyperplane under a (λ, C) continuous quasi-isometric embedding lies in a D'-neighbourhood of a hyperplane, where $D' = D'(\lambda, C)$. If $\pi\colon X \to T$ is the canonical surjection, we finally obtain the following generalisation of theorem 2.1 and corollary 2.2:

THEOREM 3.8. *Let $f\colon X \to X$ a quasi-isometric embedding. Then there is a constant $D = D(f) > 0$ with the property that for any vertex $x' \in T$ of valency at least 3, there is a vertex $y' \in T'$ such that $f(\pi^{-1}(x'))$ and $\pi^{-1}(y')$ have Haussdorf-distance at most D.*

If T has infinitely many ends, there is a well-defined and natural morphism $\theta\colon QI(G) \to QI(T)$.

Proof. The proof of theorem 2.1 goes through virtually unchanged.

If T has infinitely many ends, there is a constant $M > 0$ such that any vertex is at distance at most M from a vertex of valency at least 3. Then any quasi-isometry f of X induces a well-defined mapping $f' \colon T \to T$. The rest of the proof of corollary 2.2 goes verbatim.

Remarks Note that, unlike in corollary 2.2, here θ is not necessarily surjective (cf (??)).

Petits changements ici, e.g. je n'aime pas dire "it is easy to see" trop souvent – B

We also remark that the above result still holds when one considers two different graphs of groups and associated spaces X and X', as in theorem 2.1.

It is worth noting that computing the quasi-isometry groups of such groups is still an open problem in general, even in the special case of finitely presented, non polycyclic, abelian-by-cyclic groups studied in (??).

3.3. Uniform embeddings

In fact, one can show that the previous results still hold for slightly weaker hypotheses about the mapping.

A mapping $f \colon X \to Y$ is said to be a *uniform embedding* if it verifies the following inequalities:

(i) there exists (λ, C) such that for all $x, \tilde{x} \in X$, $d_Y(f(x), f(\tilde{x})) \leq \lambda d_X(x, \tilde{x}) + C$.

(ii) for all $A > 0$, there exists $B > 0$ such that for all $x, \tilde{x} \in X$, if $d_Y(f(x), f(\tilde{x})) \leq A$ then $d_X(x, \tilde{x}) \leq B$.

Note that the first property implies the following one:

(i)' for all $A' > 0$, there exists $B' > 0$ such that for all $x, \tilde{x} \in X$, if $d_X(x, \tilde{x}) \leq A'$ then $d_Y(f(x), f(\tilde{x})) \leq B'$.

A mapping verifiying the properties (i)' and (ii) will be called a *weak uniform embedding.*

If X is connected, then any weak uniform embedding is a uniform embedding. Note that quasi-isometric embeddings are uniform embeddings, and that any continuous uniform embedding between proper spaces is a proper mapping.

It is not difficult to show that if $f \colon X \to Y$ is an almost surjective weak uniform embedding, then there exists $M > 0$ and a weak uniform embedding $g \colon Y \to X$ such that $d_X(g \circ f(x), x) \leq M$ for all $x \in X$ and $d_Y(f \circ g(y), y) \leq M$ for all $y \in Y$. Any such g is called a weak inverse of f.

Lemma 2.3 remains true if f is a weak uniform embedding. One can see that the "connect-the-dots argument" still holds thanks to inequality (i)', so that one can assume that f is continuous. In order to show that f is proper, one only needs to show that the preimages of bounded sets are bounded, which remains true due to inequality (ii). Then, thanks to (i)', there is a constant B' such that f cannot identify two points which are more than B' away from each other, so that the rest of the proof still holds.

Similarly, the proof of lemma 2.4 can be adapted easily: let ψ be a weak inverse of φ. Then one only needs inequality (i) to apply the "connect-the-dots" argument for φ and ψ. For the rest of the proof, one only needs the facts that $\psi \circ \varphi$ is uniformly close to identity, and that φ is proper. The analogue of 2.1 follows.

4. Counterexamples

One motivation of the present paper was the connection of the main theorem with quasi-isometries of $F_2 \times F_2$: it was also proved in (??) that $QI(F_2 \times F_2) \cong (QI(F_2) \times QI(F_2)) \rtimes \mathbb{Z}_2$, where the $\mathbb{Z}_2$-factor comes from the interchange $(t', t'') \mapsto (t'', t')$. However, we shall give an example to show that this result is more subtle than our main theorem, and not directly amenable to our methods of proof. More precisely, one might conjecture that, in analogy with the previous cases, any quasi-isometric embedding $T \times \mathbb{R} \to T \times T$ sends any plane uniformly close to a product of geodesics.

rajouté "To disprove this" – B

To disprove this, we let T be the Cayley graph of the group F_2, equipped with a basepoint $*$, one of the vertices of T. Let γ' be a geodesic ray in T starting at $*$.

PROPOSITION 4.1. *There exists a quasi-isometric embedding* $g\colon T \times \mathbb{R} \to T \times T$ *such that* $g(\{*\} \times \mathbb{R}) = \gamma' \times \{*\} \cup \{*\} \times \gamma'$. *That is,* g *sends a geodesic* $\{*\} \times \mathbb{R}$ *to an* "L"*-shaped line in* $T \times T$.

Proof. Let T_+ be a tree equal to T, except that at the vertex $*$ two extra (fifth and sixth) branches (consisting of rooted infinite four-valent trees) have been attached; so we have $T \subset T_+$. Let $\gamma \subseteq (T_+ \backslash T) \cup *$ be an infinite geodesic ray in one of the added branches, starting at $*$. It is well known that there exists a quasi-isometric embedding $T_+ \to T$ that sends γ homeomorphically onto γ'. Thus in order to prove the theorem it suffices to construct a quasi-isometric embedding $f\colon T \times \mathbb{R} \to T_+ \times T_+$ satisfying the same condition $f(\{*\} \times \mathbb{R}) = \gamma \times \{*\} \cup \{*\} \times \gamma$.

We shall first construct f, and then prove that it is a quasi-isometric embedding. We define f on the 0-level $T \times \{0\}$ by $f((t,0)) = (t,t)$. On the x-level $T \times \{x\}$ ($x \in \mathbb{R}_+$), the definition goes as follows: we first define $g_x\colon T \to T_+$ to be the continuous function which moves any point in T by a distance of x towards the limit point at infinity γ^∞ of the geodesic ray γ (e.g. γ is just shunted by the distance x along itself). Now we define $f((t,x)) = (t, g_x(t))$. Similarly for $x < 0$ we define $f((t,x)) = (g_{-x}(t), t)$.

We now prove that f is a quasi-isometry, and in fact bilipschitz. Let $(t,x), (t',x') \in T \times \mathbb{R}$ be two points. Obtaining an upper bound for $d_{T_+ \times T_+}(f(t,x), f(t',x'))$ is easy – just observe that, using the triangle inequality, $d_{T\times T}(f(t,x), f(t',x')) \leq \sqrt{2} d_T(t,t') + |x - x'|$.

It remains to give a lower bound; such a bound can be immediately deduced from the following inequalities (1), (2a), and (2b):

$$d_{T_+ \times T_+}(f(t,x), f(t',x')) \geqslant \frac{1}{\sqrt{2}} |x - x'|, \qquad (1)$$

$$d_{T_+ \times T_+}(f(t,x), f(t',x')) \geqslant \sqrt{2} d_T(t,t') - |x - x'| \qquad \text{if } x \cdot x' < 0, \qquad (2a)$$

$$d_{T_+ \times T_+}(f(t,x), f(t',x')) \geqslant d_T(t,t') \qquad \text{if } x \cdot x' \geqslant 0. \qquad (2b)$$

From inequalities (2b) if $x \cdot x' \geq 0$ and (1) and (2a) if $x \cdot x' < 0$, one can deduce the following inequality:

$$d_{T_+ \times T_+}(f(t,x), f(t',x')) \geqslant \frac{\sqrt{2}}{\sqrt{2}+1} d_T(t,t') \qquad (2c)$$

To see that inequality (1) holds, note that $T_+ \times T_+$ is "foliated" by sets

$$S_x := \{(t,t') \mid d_T(t,t') = x, \text{ and } t' \text{ lies on a geodesic from } t \text{ to } \gamma^\infty\} \ (x \geqslant 0),$$

$$S_x := \{(t,t') \mid d_T(t,t') = x, \text{ and } t \text{ lies on a geodesic from } t' \text{ to } \gamma^\infty\} \ (x \leqslant 0).$$

With this definition, $f(t,x) \in S_x$ for all $(t,x) \in T \times \mathbb{R}$. We now observe that the subsets S_x and $S_{x'}$ of $T_+ \times T_+$ have distance at least $\frac{1}{\sqrt{2}}|x - x'|$ (because walking from a point (t,t') to a point at distance 1 from it, we can change the quantity $d_{T_+}(t,t')$ by at most $\sqrt{2}$).

For inequality (2a), which is of course vacuous if $d_T(t,t') < |x - x'|$, we notice that $d_{T_+ \times T_+}(f(t,0), f(t',0)) = \sqrt{2} d_T(t,t')$. Moreover, $d_{T_+ \times T_+}(f(t,0), f(t,x)) \leq |x|$, and similarly for t'; (2a) follows from the triangle inequality.

Finally for (2b) observe that the projections $p_1, p_2\colon T_+ \times T_+ \to T_+$ onto the first or second coordinate are distance decreasing functions. For $x, x' \geqslant 0$ it follows that $d_{T_+ \times T_+}(f(t,x), f(t',x')) \geqslant d_{T_+}(p_1(f(t,x)), p_1(f(t',x'))) = d_T(t,t')$. For $x, x' \leqslant 0$ one uses the same argument with p_1 replaced by p_2.

Let Y be the infinite tripod. We want to think of Y as $\mathbb{R} \cup \alpha$, with α isometric to $\mathbb{R}_+$ and the points $0 \in \mathbb{R}$ and $0 \in \mathbb{R}_+$ identified.

THEOREM 4.2. *There exists a quasi-isometric embedding* h: $T \times Y \to T \times T$ *sending the geodesic* $\{*\} \times \mathbb{R}$ *to the* L*-shaped line* $\gamma \times \{*\} \cup \{*\} \times \gamma$.

Proof. As before, it suffices to construct a quasi-isometric embedding $\tilde{g}$: $T \times Y \to T_+ \times T_+$ satisfying the condition $g(\{*\} \times \mathbb{R}) = \gamma \times \{*\} \cup \{*\} \times \gamma$.

In the previous proposition, we constructed a mapping f: $T \times \mathbb{R} \to T_+ \times T_+$. We observe that the preimage $f^{-1}(T \times \{*\})$ is the set $T_0 := \{(t, d_T(t,*)) \mid t \in T\} \subset T \times \mathbb{R}$, which is just a quasi-isometric copy of T. (Note that T_0 is only the preimage of $T \times \{*\}$, not of $T_+ \times \{*\}$.) Let ψ: $T \times \mathbb{R} \to T \times \mathbb{R}$ be defined by $\psi(t,x) = (t, x + d_T(t,*))$. Clearly ψ is a quasi-isometry which sends $T \times \{0\}$ to T_0, and $\psi(\{*\} \times \mathbb{R}) = \{*\} \times \mathbb{R}$.

We now claim that the mapping $f \circ \psi$: $T \times \mathbb{R} \to T_+ \times T_+$ can be extended to a quasi-isometric embedding $T \times Y \to T_+ \times T_+$. So we have to extend $f \circ \psi$ over $T \times \alpha$, and we shall do it in the most obvious possible way. Let δ be another geodesic ray in $T_+ \backslash T$ starting at $*$, disjoint from γ except at $*$, and $\delta(x) \in T_+$ the point on it at distance x from $*$. Then the extension of $f \circ \psi$ over $T \times \alpha$ is defined by $(t,x) \mapsto (t, \delta(x))$. One can show that this extended mapping is a quasi-isometric embedding, using similar methods as those of the previous proposition.

We finish with a few speculations about semidirect product spaces of the form $T \rtimes_\Phi \mathbb{Z}$, where Φ: $T \to T$ is a quasi-isometry (typically T is the Cayley graph, and Φ an automorphism of F_m. Such spaces have been studied in (??)). Explicitely, we take a $\mathbb{Z}$-family of copies of $T \times [0,1]$ and glue the "top end" $T \times \{1\}$ of the jth to the "bottom end" $T \times \{0\}$ of the $j+1$st, using Φ as the gluing map. This space contains some obvious "hyperplanes": their intersection with each "level", i.e. each copy of $T \times [0,1]$, consists exactly of (a geodesic in T)$\times (0,1]$, and if L_j is the geodesic on the jth level, then L_{j+1} is the unique geodesic uniformly close to $\Phi(L_j)$.

It is a major open problem to decide whether quasi-isometries of such spaces send obvious hyperplanes uniformly close to obvious hyperplanes. It is tempting to conjecture some generalisation of lemma 2.6 like: "If f: $\mathbb{R}^2 \to T \rtimes_\Phi \mathbb{Z}$ is a quasi-isometric embedding (with $\mathbb{R}^2$ carrying any proper uniformly contractible metric) then $\mathrm{im}(f)$ is at finite Hausdorff distance from an obvious hyperplane." Unfortunately, this conjecture appears to be false.

We outline here how a counterexample may be constructed; details will appear elsewhere. If Φ: $F_m \to F_m$ is a free group automorphism, lifting to a quasi-isometry of the Cayley graph T of F_m with six fixed points on the limit set, three of which are attracting (say a_1, a_3, a_5) and the other three repelling (say a_2, a_4, a_6) - this can e.g. be obtained using a pseudo-Anosov automorphism of a punctured surface). The idea is now to send a plane into $F_m \rtimes_\Phi \mathbb{Z}$ in the following way: consider six rays $\mathbb{R}_+$ emanating in a starlike fashion from the point $(0,0)$ in the plane, cutting it into six "wedges". We send this star to the 0th copy of $T_m \times [0,1]$, to six infinite rays, and we denote the limit point at infinity of the image of the ith ray of the star (in the cyclic ordering) by $b_i \in \partial T \times [0,1]$. We choose the mapping in such a way that $\lim_{j\to\infty} \Phi^j(b_{2i-1}) = \lim_{j\to\infty} \Phi^j(b_{2i}) = a_{2i-1}$ (indices modulo 6), and $\lim_{j\to-\infty} \Phi^j(b_{2i}) = \lim_{j\to-\infty} \Phi^j(b_{2i+1}) = a_{2i}$. We can now extend the mapping over the six wedges in $\mathbb{R}^2$, in such a way that 3 of them get sent to the "upper half" ($j \geqslant 0$) of $T \rtimes_\Phi \mathbb{Z}$, and the other three into the lower half ($j \leqslant 0$). More precisely, this is done in such a way that, restricting our attention to say the upper half of $T \rtimes_\Phi \mathbb{Z}$, the image of the plane just seems to consist of 3 obvious hyperplanes, namely the ones whose intersection with the 0-level are the geodesics $\overline{b_1 b_2}$, $\overline{b_3 b_4}$, and $\overline{b_5 b_6}$.

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