3 The singular cubical set of a topological space

Abstract For any topological space X let $C(X)$ be the realisation of the singular cubical set of X; let $*$ be the topological space consisting of one point. In [1] Antolini proves, as a corollary to a general theorem about cubical sets, that $C(X)$ and $X \times C(*)$ are homotopy equivalent, provided X is a CW-complex. In this chapter we give a short geometric proof that for *any* topological space X there is a natural weak homotopy equivalence between $C(X)$ and $X \times C(*)$. This chapter is joint work with Rosa Antolini.

Let X be a topological space. The realisation of the singular cubical set of X, $C(X)$, is the CW-complex defined as follows: let $I := [0,1]$, with basepoint at 0 and oriented from 0 to 1. So I^n is the n-cube, equipped with a basepoint, an ordering of the coordinate directions and an orientation of all its edges. Then the n-cells of $C(X)$ are given by copies of the n-cube I^n, one for each continuous map from I^n to X. A typical n-cell of $C(X)$ is denoted by I^n_f, where $f\colon I^n \to X$.

To define the attaching maps consider the face maps

$$\delta_i^\epsilon(n)\colon I^n \to I^{n+1}, \text{ given by } (x_1, ..., x_n) \mapsto (x_1, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_n)$$

for $i = 1, \ldots, n+1$ and $\epsilon \in \{0,1\}$. The attaching maps of $C(X)$ are defined by identifying the point $(x_1, ..., x_{i-1}, \epsilon, x_i, ..., x_n) \in \partial I^{n+1}_f$ with the point $(x_1, ..., x_n) \in I^n_{f \circ \partial_i^\epsilon(n)}$. For background information on cubical sets see [1] or [6].

Remark $C(*)$ has only one cell in each dimension, denoted by I^n_*. It is proved in [7] that $C(*)$ is homeomorphic to S^1_∞, the free topological monoid on the circle. By a theorem of James [15] this is homotopy equivalent to ΩS^2, the loop space on the sphere. So we have $C(*) \simeq \Omega S^2$.

There is a natural map $\phi\colon C(X) \to X \times C(*)$ defined in the following way: the constant map $X \to *$ induces $p\colon C(X) \to C(*)$. Furthermore, each point $x \in C(X)$ is in the interior of a unique cell, say I^n_f, of $C(X)$, where $n \in \mathbf{N}$ and $f\colon I^n \to X$. Then we define $\phi(x) := (f(x), p(x)) \subseteq X \times C(*)$.

Theorem 3.1 *Let X be a topological space. Then the map $\phi\colon C(X) \to X \times C(*)$ is a weak homotopy equivalence, i.e. $\phi_*\colon \pi_q(C(X)) \to \pi_q(X \times C(*)) \cong \pi_q(X) \oplus \pi_q(C(*))$ is an isomorphism for all $q \in \mathbf{N}$.*

We start by reviewing the correspondence between maps from a manifold to $C(X)$ and certain diagrams in the manifold, which was introduced in [7]. Let Q^q be a q-dimensional smooth manifold. A map $f\colon Q^q \to C(X)$ is called transverse if for each open i-cell c of $C(X)$ there exists a $q-i$-dimensional manifold M^{q-i} with boundary and a diffeomorphism $\kappa\colon f^{-1}(c) \to M^{q-i} \times c$ such that the diagram

$$\begin{array}{ccc} f^{-1}(c) & \xrightarrow{\kappa} & M^{q-i} \times c \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_2} \\ & & c \end{array}$$

commutes, where p_2 denotes the projection on the second coordinate. Thus M^{q-i} is a submanifold of Q^q, which is framed by copies of c.

By indicating which point of Q^q is mapped to which cell of $C(X)$ we obtain a *diagram* in Q^q (see Figure 3.1). We call the closure of a path component of the preimage of an open cell in the complex $C(X)$ a *region* of the diagram. As seen above, a region which is the preimage of an i-cell is naturally diffeomorphic to $M^{q-i} \times I^i$; using the notation of handle theory we can call all the manifolds $M^{q-i} \times \{d\}$ and $\{m\} \times I^i$ ($d \in I^i$, $m \in M^{q-i}$) parallels of the core respectively the cocore of $M^{q-i} \times I^i$, and we can call the region a $q-i$-region or a region of index $q-i$. For instance, in Figure 3.1(a) we have a diagram representing a map $S^2 \to C(*)$ with 3 0-regions, 6 1-regions and 5 2-regions. The ordering of the coordinate directions of the 2-cell of $C(*)$ has been indicated by drawing the preimages of $I \times \{0,1\} \subseteq I^2$ in solid, and the preimages of $\{0,1\} \times I \subseteq I^2$ in broken lines.

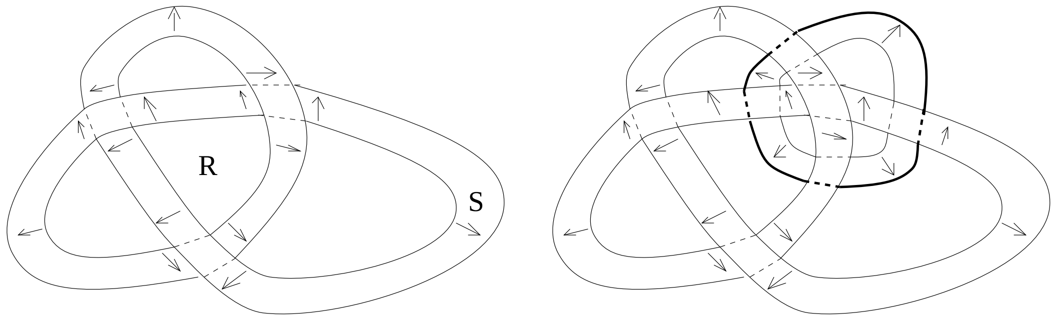

Figure 3.1: (a) A diagram in S^2 with 14 regions. (b) introducing a sphere

Two regions in a diagram are called *adjacent* if they have at least one point in common. For instance, there are 6 regions adjacent to the region R in Figure 3.1(a). A region is called *adjacent to the core of a region* $M^{q-i} \times I^i$ if it has at least one point in common with $\partial M^{q-i} \times \operatorname{int}(I^i)$. For instance, there are two regions, both of index 0, adjacent to the core of the region S.

Thus in a diagram every region is labelled by one cell of $C(X)$, and the labelling of adjacent regions is consistent in the following sense: if a $q-i$-region is labelled by a cell c then an adjacent $q-i+1$-region is labelled by a face of c.

To prove Theorem 3.1 we will exploit the fact that the above construction defines a natural one-to-one correspondence between transverse maps from a manifold Q^q to $C(*)$ and diagrams in Q^q with a consistent labelling of the regions by cells of $C(*)$. Homotopies of maps $Q^q \to C(*)$ correspond to cobordisms of such diagrams.

The ways in which a diagram can change under a (transverse) homotopy of the map $Q^q \to C(*)$ are explained in detail in [7], chapter 4. Here we only need the following procedure: let $B \subseteq Q^q$ be a q-ball such that ∂B intersects the diagram transversely. Let $\Gamma \cong S^{q-1} \times [0,1]$ be a collar of ∂B in B. Then we define a homotopy f_t $(t \in [0,1])$ of $f = f_0: Q^q \to C(*)$ which is fixed on $Q^q - B$ as follows: if for a point $q \in B - \Gamma$ we have that $f_0(q) = (x_1, ..., x_i) \in I^i$ (in the unique i-cube of $C(*)$), then we define $f_t(q) := (x_1, ..., x_i, t) \in I^{i+1}$. If for a point $q = (s, u) \in \Gamma \cong S^{q-1} \times [0,1]$ we have $f_0(q) = (x_1, ..., x_i) \in I^i$, then we define $f_t(q) := (x_1, ..., x_i, ut) \in I^{i+1}$. The effect of this homotopy is to 'introduce a sphere inside ∂B', and to leave the rest of the diagram unaffected. (By making the homotopy transverse we obtain a cobordism of diagrams, i.e. a diagram in $Q^q \times I$. It is just the trivial cobordism, with one extra dome, whose boundary is $\Gamma \subseteq Q^q \times \{1\}$.) An example is given in Figure 3.1(b), where the bold line denotes ∂B. By placing the collar of B *outside* B, we can in a similar fashion 'introduce a sphere outside ∂B'.

Similarly, homotopy classes of maps $Q^q \to C(X)$ are in one-to-one correspondence with cobordism classes of diagrams in Q^q with a consistent labelling of the regions by cells of $C(X)$, i.e. maps $I^n \to X$.

Proof of the Theorem Let Q^q be a smooth q-manifold, possibly with boundary. For every map $s : Q^q \to X \times C(*)$, there exist two maps $s_X: Q^q \to X$ and $s_C: Q^q \to C(*)$ such that $s(q) = (s_X(q), s_C(q))$ for all $q \in Q^q$. If the map s_C is transverse, then it gives rise to a diagram in Q^q. We now describe a necessary and sufficient condition for the existence of a map $\tilde{s}: Q^q \to C(X)$ such that $s = \phi \circ \tilde{s}$. If for every single $(q-i)$-region of the diagram ($i \in \{0, ..., q\}$) we have that

$$s_X((m_1, d)) = s_X((m_2, d)) \ \text{ for all } \ m_1, m_2 \in M^{q-i}, \ d \in I^i,$$

i.e. if all parallels of the cocore are mapped to X in the same way, then s_X together with the diagram determines a unique lifting $Q^q \to C(X)$, by sending a cocore c to the cell $I^i_{s_X|_c}$. Conversely, for any map $\tilde{s}: Q^q \to C(X)$ the diagram of $\phi \circ \tilde{s}: Q^q \to C(X) \to$

$X \times C(*)$ satisfies the above condition. We say 's_X is compatible with the diagram of s_C' or simply 's is compatible'. For instance, for any compatible map s the restriction of s_X to a q-region is a constant map, while the restriction to a 0-region is arbitrary.

Proof that ϕ_* is surjective: Let $s\colon S^q \to X \times C(*)$ be a representative of a given element of $\pi_q(X \times C(*))$. We can make s_C transverse (this is e.g. proved in [7]). Then we introduce q spheres inside one q-region of the diagram of s_C, which intersect each other as indicated (for the case $q = 2$) in Figure 3.2. Thus we can assume that the diagram of s_C has at least one region R of index 0. After a homotopy of s_X we can assume that $s_X(t) = s_X(*_{S^q})$ for all $t \in S^q - R$ (where $*_{S^q}$ is the basepoint of S^q). Now s_X is compatible with the diagram of s_C, so s lifts to a map $\tilde{s}\colon S^q \to C(X)$. □

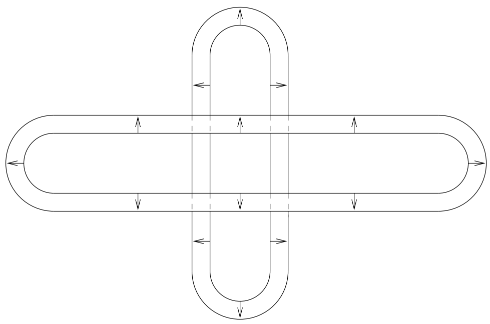

Figure 3.2: A diagram in S^2 with 4 regions of index 0

Proof that ϕ_* is injective: Let B^q be a q-dimensional ball, and let $s\colon B^q \to X \times C(*)$ be a map such that s_C is transverse, and such that the restriction $s|_{\partial B^q}\colon S^{q-1} \to X \times C(*)$ is compatible. Our aim is to homotope s relatively to ∂B^q such that s is compatible everywhere.

After collaring we can assume that there exists an embedded q-ball $B \subseteq B^q$ such that the restriction $s|_{B^q - B}$ is compatible. Next we introduce a sphere *inside* ∂B in the diagram of s_C (see Figure 3.3). Still, $s|_{B^q-B}$ is compatible, and in addition we have that every region in B^q either lies entirely in B or entirely in $B^q - B$. For the rest of the proof we shall leave $s_X|_{B^q-B}$ untouched. The following two observations will be crucial in the proof:

Observation 1 Given a map $f\colon B^q \to X \times C(*)$ such that f_C is transverse, and a subset U of B^q such that $f|_U$ is compatible. By a homotopy of f_C introduce a sphere into the diagram, in any position. Then after the homotopy $f|_U$ is still compatible.

Observation 2 Suppose we have a map $B^q \to C(*) \times X$, a ball $B \subseteq B^q$ and a thickened sphere in B inside ∂B as above. Let U be a region in B. Then any region adjacent to the core of U is also in B.

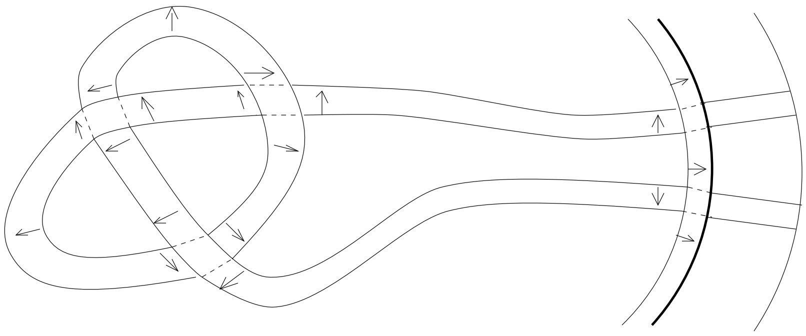

Figure 3.3: The bold line is ∂B.

Recall that we want to modify s relatively to ∂B^q such that it becomes compatible on all B^q. We shall do this by downward induction on the index of the regions. So suppose inductively that we have a transverse map $s\colon B^q \to X \times C(*)$ such that s restricted to any region of index at least $q-i+1$ is compatible. To do the induction step $i-1 \to i$, we shall exhibit a homotopy of s which leaves $s|_{\partial B^q}$ and $s_X|_{B^q - B}$ unaltered, such that s restricted to any $q-i$-region is compatible.

Let $R_1, .., R_n \subseteq B$ be the $q-i$-regions in B. Each of them is a q-manifold with boundary, and its core is a $q-i$-dimensional manifold with boundary. The homotopy will only change s_X on $R_1, ..., R_n$ and the regions adjacent to their cores. Let $\{B_j\}_{j\in\{1,...,J\}}$, where $J \in \mathbf{N}$, be a set of q-balls in B^q such that

(1) for each $j \in \{1, ..., J\}$ the intersection of B_j with the union of all cores has at least one path component which is homeomorphic to a $q-i$-ball. We call this component A_j. Note that the intersection of B_j with the corresponding $q-i$-region is homeomorphic to $A_j \times I^i$.

(2) $\bigcup_{j=1,...,J} A_j \times I^i = R_1 \cup ... \cup R_n$.

It is clear that such a set of balls exists. Moreover, we have that for $R_l = M^{q-i} \times I^i$ ($l = 1, ..., n$) the map s, restricted to $M^{q-i} \times \partial I^i$, is compatible, i.e. that all parallels of the boundaries of the cocores are mapped to X in the same way.

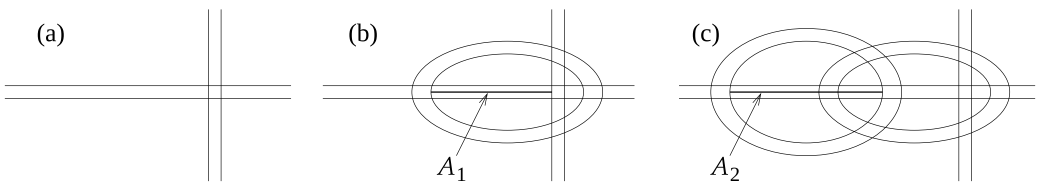

Figure 3.4: The induction on j ($q = 2$, $i = 1$)

We now use a second induction, on the index j of the balls B_j ($j = 1, ..., J$). The inductive hypothesis is that s is compatible on those regions in $A_k \times I^i$ ($k = 1, ..., j-1$) which have index $q - i$. For $j = 1$, this is trivially satisfied. To do the step from $j-1$ to j, introduce a $q-1$-sphere outside ∂B_j. This will create new $q-i$-regions outside R^{q-i}, but, by Observation 1, s is compatible on these newly created regions. Let m be a point in the interior of A_j. Recall that A_j is a $q-i$-ball, so we can radially expand the cocore $m \times I^i$ until it fills $A_j \times I^i$. This induces a homotopy of the map s_X

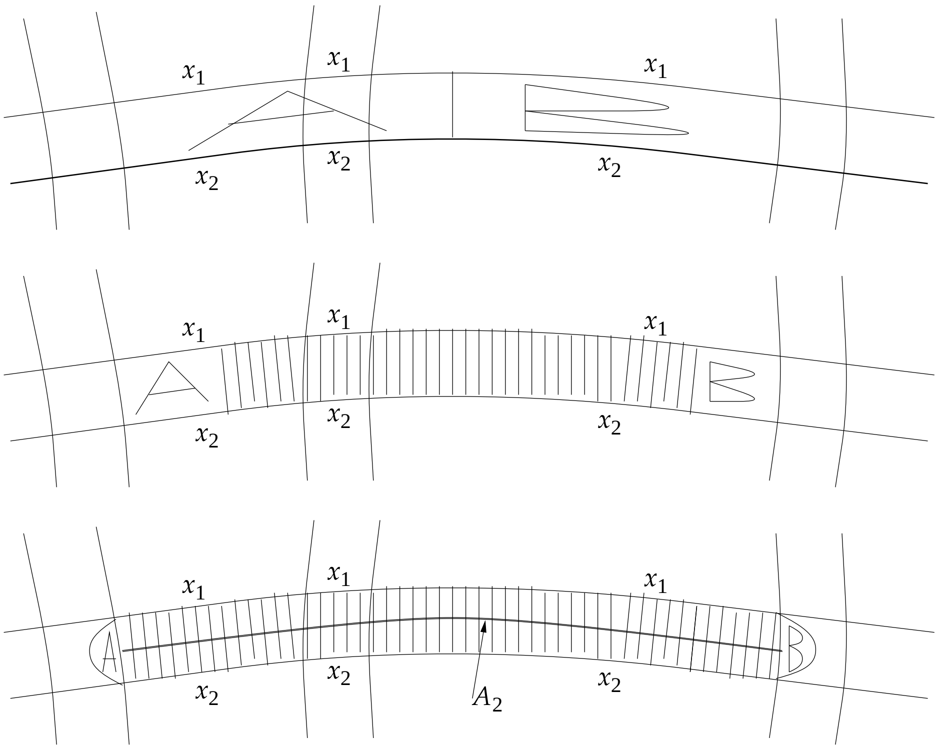

Figure 3.5: The homotopy of the map s_X in Figure 3.4(c) ($x_1, x_2 \in X$)

which changes s_X only on $A_j \times \text{int}(I^i)$ and on regions adjacent to the cores of regions in $A_j \times I^i$. In particular, $s_X|_{B^q - B}$ remains unaltered, by Observation 2. It also follows that for $R_l = M^{q-i} \times I^i$ ($l = 1, ..., n$) the map s, restricted to $M^{q-i} \times \partial I^i$, is still compatible. Now s is compatible on $A_j \times I^i$, and regions adjacent to the cores of the regions in $A_j \times I^i$ have index less then $q - i$, so the inductive hypothesis is satisfied, with $j - 1$ replaced by j. The homotopy is illustrated in Figure 3.5.

This completes the induction step on j. Since j only runs through a finite number of values, this also finishes the induction step on i, and thus proves the theorem. ☐

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